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Discrete Mean Values of Dirichlet L-functions

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Abstract

In 1911 Landau proved an asymptotic formula for sums of the form $\sum_{\gamma \le T} x^\rho$ over the imaginary parts of the nontrivial zeros of the Riemann zeta function. The formula provided yet another deep connection between $\zeta(s)$ and the prime numbers—the asymptotic growth increases from a $O(\log T)$ term to T if x is a prime power. Gonek extended this to a uniform estimate in x in his 1993 paper. He then applied the result to prove a discrete mean value estimate for $\zeta(s)$. Inspired by this proof we will look at discrete mean values of two Dirichlet L-functions and, in particular, show that for a positive proportion of the zeros of $\zeta(s)$, these L-functions are linearly independent over $\mathbb{R}$ for primitive non-principal real characters modulo distinct primes.

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1 Introduction

Mean values arise naturally in analytic number theory. This is because the most important arithmetic functions are not smooth, so it makes sense to consider their averages to better understand their behaviour. These smoothed out averages are then easier to work with than the original functions. In the discrete case the idea is to study the sum of the values of a function within some interval divided by the length of that interval. Formally, for a function $f:\mathbb{N} \longrightarrow \mathbb{C}$ we say its discrete mean value is $g:\mathbb{R} \longrightarrow \mathbb{C}$ as $x \longrightarrow \infty$ if

$$\sum_{n\leq x} f(n) = xg(x) + o(xg(x)),$$

where $g(x)$ is monotonic increasing (see section 2.1 for the little-o notation) [14, p. 10]. For example, let $\pi(x)$ denote the number of primes less than or equal to x. This is clearly an increasing step function expressible as the sum $\sum_{n\leq x} u_{\mathbb{P}}(n)$, where $u_{\mathbb{P}}(n) = 1$ if n is a prime and 0 otherwise. Now, the Prime Number Theorem says that

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1.$$

From this we then know that the mean value of $u_{\mathbb{P}}(x)$ is $\log^{-1} x$. This also works for functions which have even wilder fluctuations than $u_{\mathbb{P}}(x)$. Let us consider for example the well-known Euler totient function $\varphi(n)$, which counts the number of invertible residue classes modulo n. Some values of this function are depicted below.

n	3	4	5	6	7	100	101	102
$\varphi(n)$	2	2	4	2	6	40	100	32

It is an easy exercise in elementary number theory to show that

$$\frac{\varphi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d},$$

where $\mu(d)$ is 1 at 1, $(-1)^k$ if $d = p_1 \dots p_k$ for distinct primes p_i, and 0 otherwise. After a little work this allows us to deduce that the discrete mean value of $\varphi(x)$ is cx for some positive constant c [14, p. 11].

It is important to notice that in order to calculate a discrete mean value for a function we have to consider its values on some countable set—the natural numbers in the above, for example. If we are actually working with a function $f:\mathbb{C} \longrightarrow \mathbb{C}$ then it is less clear which points on the complex plane we should choose to average over. In such cases it might be more convenient to work with continuous mean values, which are of the form

$$\frac{1}{T}\int_1^T |f(\sigma + it)| \,\mathrm{d}t,$$

for some $\sigma \in \mathbb{R}$, for example. Taking the integral smooths out the behaviour of the function even further, which means that sudden spikes in the value distribution of the function contribute less. In practice, we often find that continuous mean values produce results easier. The disadvantage is that—unlike with discrete mean values—we are no longer restricting the function $f(z)$ to specific points of interest, but to an interval instead.

Mean values are also closely related to central problems in analytic number theory. One of the most important unproven conjectures in mathematics is the Riemann Hypothesis (RH). It originated in Bernhard Riemann's paper "Über die Anzahl der Primzahlen unter einer gegebenen Grösse" [17] in 1859. The conjecture states that the nontrivial zeros of the Riemann zeta function, $\zeta(s)$, lie on the line $\Re(s) = 1/2$. We will give a precise statement in section 2.4. Often, by assuming the RH, it is possible to obtain optimal estimates as well as to gain insight into number-theoretic problems—as is the case in this paper. Therefore a great deal of effort has been put into trying to resolve the validity of the RH, so far without success.

Since 1895 the RH has been generalised in numerous ways by considering, for example, other Dirichlet series. Various equivalent forms and corollaries of the RH have been found as well. One such corollary, which is of particular interest to us, is the Lindelöf Hypothesis (LH), named after the Finnish mathematician Ernst Lindelöf. It says that for any $\epsilon > 0$ we have

$$\zeta(\tfrac{1}{2} + it) = O_\epsilon(t^\epsilon),$$

as $t \longrightarrow \infty$. Again, see section 2.1 for the big-O notation. The subscript ϵ means that the implied constant depends on ϵ. The LH—like the RH—has resisted all attempts at a proof. One important line of attack is by using continuous mean values. More precisely, the LH is equivalent to showing that

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,\mathrm{d}t = O_\epsilon(t^\epsilon)$$

for all $\epsilon > 0$ and $k \geq 1$, $k \in \mathbb{N}$ [21, p. 328]. So, given enough knowledge about the mean values of the Riemann zeta function on the critical line (or equivalently in the region $1/2 < \sigma < 1$) we can prove the Lindelöf Hypothesis, which in turn gives more evidence that the RH should be true. However, computing even some of these mean values has proven to be extremely difficult. This is partly due to the fact that the series representation of $\zeta(s)$ is not even valid in this region, nor is the functional equation (see section 2.4) very helpful. Instead we have to resort to more complicated tools, such as the approximate functional equation which will be introduced later. So far only the second and fourth moments of the Riemann zeta function have been established. The second moment, found by Hardy and Littlewood in 1918 [12, pp. 203-204], is

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2}\,\mathrm{d}t \sim \log T.$$

It is interesting to compare this with the well-known estimate away from the critical line:

$$\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^{2}\,\mathrm{d}t \sim \zeta(2\sigma)$$

for $\sigma > 1/2$ [21, p. 140]. It can be seen that $\zeta(s)$ grows (on average) more on the critical line than to the right of it. This is in line with the assumption that the nontrivial zeros should lie on the critical line. It took almost 80 years to reach a plausible conjecture for the remaining moments. In 2000, Keating and Snaith [9] used methods of random matrix theory to hypothesise a general formula.

In this paper we shall focus on discrete mean values for a class of functions arising from Dirichlet L-series extended to all of $\mathbb{C}$. This extension gives rise to so-called Dirichlet L-functions. Being generalisations of the Riemann zeta function, these L-functions satisfy properties closely resembling those of $\zeta(s)$. It is this connection which inspires us to follow Gonek's proof [5] of a discrete mean value result for the zeta function, as well as Petridis' proof of Theorem 1.9 [16] for GL_2 L-functions, and imitate it with Dirichlet L-functions. We shall prove the following:

Main Theorem *Assume the Riemann Hypothesis. Let χ_1, χ_2 be real, primitive, and non-principal characters mod q and l, respectively, where $q \neq l$ are primes. Then, for a positive proportion of the nontrivial zeros of $\zeta(s)$, for a fixed*

$\sigma \in (\frac{5}{6}, 1)$, *the two Dirichlet* L*-functions* $L(\sigma + i\gamma, \chi_1)$ *and* $L(\sigma + i\gamma, \chi_2)$ *are linearly independent over* $\mathbb{R}$.

Here γ denotes the imaginary part of a nontrivial zero of the Riemann zeta function. This result fits naturally within the current developments in the theory of Dirichlet L-functions. In 1976 Fujii [3] showed that a positive proportion of zeros of $L(s, \psi)L(s, \chi)$ are distinct, where the characters are primitive and not necessarily of distinct moduli. A zero of the product is said to be distinct if it is a zero of only one of the two, or if it is a zero of both then it occurs with different multiplicities for each function. It is, in fact, believed that all zeros of Dirichlet L-functions to primitive characters are simple and that two L-functions with distinct primitive characters do not share any nontrivial zeros at all. This comes from the Grand Simplicity Hypothesis (GSH) [18]. It states that the set

$$\{\gamma : L(\tfrac{1}{2} + i\gamma, \chi) = 0 \text{ and } \chi \text{ is primitive}\}$$

is linearly independent over $\mathbb{Q}$. Since we are counting with multiplicities, it is implicit in the statement of the GSH that all zeros of Dirichlet L-functions are simple, and that $\gamma \neq 0$, i.e. $L(\frac{1}{2}, \chi) \neq 0$. A similar result is expected for an even bigger class of functions. Murty [13] quotes his brother and his result from 1994 that two functions of the Selberg class[1] $\mathcal{S}$ cannot share too many zeros (counted with multiplicity). They show that if F, $G \in \mathcal{S}$ then $F = G$ provided that

$$|Z_F(T) \Delta Z_G(T)| = o(T),$$

where $Z_F(T)$ denotes the set of zeros of $F(s)$ in the region $\Re(s) \geq 1/2$ and $|\Im(s)| \leq T$, and Δ is the symmetric difference. Returning to Dirichlet L-functions, Garunkštis, Kalpokas, and Steuding [4] proved an asymptotic formula for a discrete mean value of a Dirichlet L-function over the zeros of another L-function. Their result is as follows:

Theorem *Let* A *and* B *be positive constants. Let* $\psi \mod Q$ *and* $\chi \mod q$ *be primitive Dirichlet characters and* $\chi \neq \psi$. *Then, uniformly for* $Q \ll \log^A T$ *and* $q \ll \log^B T$, *we have*

[1] The Selberg class is a class of functions with properties similar to those of $\zeta(s)$, such as a Dirichlet series and Euler product representation for $\Re(s) > 1$, and a functional equation. In fact, ζ, $L(s, \chi) \in \mathcal{S}$.

$$\sum_{0<\gamma_\chi\le T} L(\rho_\chi,\psi) = \frac{T}{2\pi}\log\frac{Tq}{2\pi e} - \delta(q,Q)L(1,\chi\overline{\psi})\psi(-1)\tau(\psi)\frac{\tau(\overline{\chi}\psi_0)}{\phi(Q)}\frac{T}{2\pi} + \frac{L'}{L}(1,\psi\overline{\chi})\frac{T}{2\pi} + O\Big(T\exp(-c\log^{\frac{1}{4}-\epsilon}T)\Big),$$

where $\delta(q,Q)=1$ if $q\mid Q$, $\delta(q,Q)=0$ otherwise, ψ_0 is the principal Dirichlet character $\mod Q$ and c is a positive absolute constant.

Under the Generalised Riemann Hypothesis (GRH) the error term can be replaced by $O\Big((TQ)^{1/2+\epsilon}q^\epsilon\Big)$, which is valid uniformly for q, $Q \ll T^{1-\epsilon}$.

Compared to Fujii's result, this theorem tells us more precisely that two Dirichlet L-functions cannot share many zeros. Our main theorem is a natural extension of these ideas. Instead of just considering the points at which an L-function has a zero, we are looking at any possible values of it. We suspect that two distinct L-functions to primitive characters tend to assume distinct values at any point. We are, however, considering the stronger statement that the values are, in fact, linearly independent over $\mathbb{R}$. Whilst we only consider a very limited set of points, based on the above discussion it is reasonable to expect that there should exist a more general version of our theorem, that is: two Dirichlet L-functions with primitive characters are linearly independent over $\mathbb{R}$ everywhere.

2 Definitions and Preliminary Results

Before beginning the main discussion we will give a list of all basic definitions and properties that we will quote in this paper without full proof. Throughout this paper we will mostly use the notation $s=\sigma+it$, where σ, $t\in\mathbb{R}$, for a complex variable as this is the standard in analytic number theory. We reserve p to always denote a fixed prime and sums (or products) of the form $\sum_p$ mean that the summation is taken over all primes in an increasing order. Also let $p^\alpha \parallel n$ signify that α is the highest power of p which divides n.

2.1 The O-notation

To help us deal with asymptotic equations we will use Landau's big-O and little-o notation defined as follows.

Definition 1 We write

$$f(x) = O(g(x))$$

if there are two positive constants M and N such that $|f(x)| \leq Mg(x)$ for all $x > N$. If there is an explicit dependence between some variable and the constant M then we may write $f(x) = O_\epsilon(g(x)^\epsilon)$, for example. Instead of the big-O we can use an equivalent notation and write

$$f(x) \ll g(x).$$

A stronger condition is implied by the little-o:

$$f(x) = o(g(x)) \iff \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

The final piece of notation is for when two functions behave in the same way asymptotically. We denote this by

$$f(x) \sim g(x) \iff \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

Notice that as we often work with complex valued functions, we may write for example that $f(z) = O(z)$ to mean $f(z) = O(|z|)$.

2.2 Methods of Multiplicative Number Theory

2.2.1 Arithmetic Functions

Multiplicative number theory studies the factorisation of numbers into primes. This is done through objects known as arithmetic functions. An arithmetic function is simply a function $a : \mathbb{N} \longrightarrow \mathbb{C}$. We say that the arithmetic function is multiplicative if $a(mn) = a(m)a(n)$ whenever m and n are coprime. Furthermore, if the same relation is true for all $m, n \in \mathbb{N}$, we say that a is completely multiplicative. Also, it is customary to denote the sum of values of an arithmetic function by a capital letter. So we write $A(x) = \sum_{n\leq x} a(n)$, for example. Let us now define some arithmetic functions which will get used later on.

Definition 2 The divisor function d is defined by

$$d(n) = \sum_{k|n} 1.$$

So in other words the divisor function just counts the number of positive divisors of n. This is clearly a multiplicative function, but not completely multiplicative as, for example, $d(4) = 3 \neq 2 \cdot 2 = d(2)d(2)$. It is not difficult to show that

$$d(n) \ll_\epsilon n^\epsilon \tag{1}$$

for any $\epsilon > 0$ [14, Exercise 1.3.2]. Another arithmetic function which we will encounter is the von Mangoldt function.

Definition 3 The von Mangoldt function is defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m,\ m > 0 \\ 0 & \text{otherwise.} \end{cases}$$

This function singles out all prime powers, but does not grow too rapidly. Even though $\Lambda(n)$ is not even multiplicative, it is still very useful in controlling the asymptotic behaviour of sums, for example. Since $\log x \ll_\epsilon x^\epsilon$ we have

$$\Lambda(n) \ll_\epsilon n^\epsilon, \tag{2}$$

for any $\epsilon > 0$. There is also a version of the Prime Number Theorem expressed as a sum of this function. It says that there is a constant $c > 0$ such that

$$\sum_{n<X} \Lambda(n) = X + O\Big(Xe^{-c\sqrt{\log X}}\Big)$$

as $X \longrightarrow \infty$. This is a very applicable estimate, especially if we use the following reformulation.

$$\sum_{n<X} \Lambda(n) = X + O\left(\frac{X}{\log^N X}\right) \tag{3}$$

for any $N > 0$ as $X \longrightarrow \infty$.

2.2.2 Summation Techniques

We often find ourselves in need of estimating sums or integrals of functions. Apart from trivial estimates, one basic method is to simply compare a sum with the corresponding Riemann sums to obtain

$$\sum_{n>x} f(n) \leq \int_x^\infty f(t)\,\mathrm{d}t + f(x),$$

for some positive decreasing $f(x)$, for example. A more advanced technique is that of partial summation which is also known as summation by parts or Abel summation. There is also a discrete version of this method which also goes by the same name. However, we will not use it. We will state the continuous version as a theorem [8, Theorem 1.3.5].

Theorem 1 (Partial Summation) *Let $x > y$ and let f be a complex valued function with a continuous derivative on $[y, x]$. Then*

$$\sum_{y<n\leq x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)\,\mathrm{d}t.$$

The proof of this result is just an exercise in algebraic manipulation. It turns out that we find ourselves working with sums where the summation does not run over natural numbers. For example, many of our sums will be over the imaginary parts γ of the nontrivial Riemann zeros ρ. For such sums the above formula is of course not valid as is, but we can still prove a similar result through a Stieltjes integral. Suppose we are considering the following sum

$$\sum_{0<\lambda\leq T} a(\lambda)f(\lambda),$$

where λ are part of some sequence of strictly increasing real numbers λ_n. Suppose also that f is continuous on $[0, T]$, and that the largest n such that $\lambda_n < T$ is $K > 0$. Let $\alpha(x) = \sum_{n=1}^\infty a(\lambda_n)I(x - \lambda_n)$, where $I(x)$ is the indicator function on $(0, \infty)$. We then apply Theorem 6.16 in [19] in order to write the sum as an integral:

$$\begin{aligned}\sum_{0<\lambda\leq T} a(\lambda)f(\lambda) &= \sum_{n=1}^{K} a(\lambda_n)f(\lambda_n)\\ &= \sum_{n=1}^{\infty} a(\lambda_n)I(T-\lambda_n)f(\lambda_n)\\ &= \int_0^T f\,\mathrm{d}\alpha\\ &= f(T)\alpha(T) - f(0)\alpha(0) - \int_0^T \alpha(x)f'(x)\,\mathrm{d}x,\end{aligned}$$

where in the last equality we have used Exercise 17 in [19].

As an example of this technique let us estimate the sum $\sum_{m\leq X} m^{-1}$, where $X > 1$. We find that

$$\begin{aligned}\sum_{m\leq X}\frac{1}{m} = \frac{1}{X}\sum_{m\leq X} 1 - 1 + \int_1^X \frac{1}{t^2}\sum_{m\leq t} 1\,\mathrm{d}t\\ = \frac{[X]}{X} - 1 + \int_1^X \frac{[t]}{t^2}\,\mathrm{d}t.\end{aligned}$$

Next we use the fact that $x = [x] + \{x\}$, and $\{x\} = O(1)$ to obtain that

$$\begin{aligned}\frac{[X]}{X} - 1 + \int_1^X \frac{[t]}{t^2}\,\mathrm{d}t &= \frac{X-\{X\}}{X} - 1 + \int_1^X \frac{t-\{t\}}{t^2}\,\mathrm{d}t\\ &= -\frac{\{X\}}{X} + \log X + \int_1^X \frac{\{t\}}{t^2}\,\mathrm{d}t.\end{aligned}$$

Now, notice that the first term is $O\left(X^{-1}\right)$ which is $O(1)$. Similarly the integral is $O(1)$. Thus we obtain that

$$\sum_{m\leq X}\frac{1}{m} = \log X + O(1).$$

We will use this result repeatedly without explicitly referring back to here.

2.3 Gamma Function

It is not uncommon for the Gamma function to pop up in unexpected places. Therefore, it is no surprise that it is closely related to the Riemann zeta function as well, mainly through the functional equation for $\zeta(s)$.

Definition 4 The Gamma function is defined for $\Re(s) > 0$ by

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, \mathrm{d}x.$$

It is possible to show by a simple argument in complex analysis that $\Gamma(s)$ converges absolutely in this region. Then, integrating by parts we can see that $\Gamma(s)$ satisfies the functional equation

$$s\Gamma(s) = \Gamma(s+1).$$

It follows from this that $\Gamma(s)$ can be extended to all of $\mathbb{C}$ as a meromorphic function with simple pole at 0 and the negative integers with residue $(-1)^m/m!$ at $-m$. See [20, p. 160] for proofs.

The Gamma function also satisfies a useful estimate known as Stirling's Formula.

Lemma 1 (Stirling's Formula) For any $\epsilon > 0$ and $s \in \{z \in \mathbb{C} : -\pi + \epsilon < \arg z < \pi - \epsilon\}$ we have

$$\log\Gamma(s) = \left(s - \frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi + O\Big(|s|^{-1}\Big), \tag{4}$$

where the implied constant depends only on ϵ [15, A4.8].

A more useful version of this formula for us comes from considering the asymptotic behaviour of $\Gamma(s)$ on fixed vertical lines. It is easy to deduce the following estimate from Stirling's Formula [1, p. 257, 6.1.45].

Lemma 2 (Asymptotical Stirling's Formula) For a fixed $x \in \mathbb{R}$, and $y \in \mathbb{R}$ such that $|y| \longrightarrow \infty$ we have

$$|\Gamma(x+iy)| \sim \sqrt{2\pi}e^{-\pi|y|/2}|y|^{x-1/2}. \tag{5}$$

Finally we need to estimate the logarithmic derivative of the Gamma function. This is also known as the digamma function and denoted by $\psi(s)$. It follows from Stirling's Formula that

$$\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)} = \log s - \frac{1}{2s} + O\Big(s^{-2}\Big). \tag{6}$$

2.4 Riemann Zeta Function

The Riemann zeta function has its origins in the work of Euler, who also introduced the current notation. He studied sums of the form

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots .$$

He was the first to find, for example, the exact sum of the above series, which is $\pi^2/6$. Thus he was led to examine the sum as a function of the exponent, which began the study of the Riemann zeta function. Sometimes we will also refer to this function just by "the zeta function" as we will not be considering more general zeta functions. We will now give a proper definition.

Definition 5 The Riemann zeta function is defined for $\Re(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

By Cauchy's integral test this series converges absolutely in its region of definition. It is also bounded and thus converges uniformly. This means that $\zeta(s)$ extends to an analytic function on the above half-plane. Moreover, in this region it satisfies the Euler product formula,

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}$$

over all primes p [8, Theorem 2.1.3]. It is immediate from the above formula that $\zeta(s) \neq 0$ in the region of absolute convergence.

Let us first attempt to differentiate this function. It is convenient to work with the logarithmic derivative of the Euler Product expression. We obtain

$$\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\sum_p (1 - p^{-s}) \frac{\mathrm{d}}{\mathrm{d}s} (1 - p^{-s})^{-1} \\
&= \sum_p p^{-s} (1 - p^{-s})^{-1} \log p \\
&= \sum_p p^{-s} \log p \sum_{n=0}^{\infty} p^{-ns} \\
&= \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}. \qquad (7)
\end{aligned}$$

Riemann showed in his seminal paper [17] in 1859 that $\zeta(s)$ satisfies the following functional equation:

$$\zeta(1-s) = \pi^{\frac{1}{2}-s}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\zeta(s), \tag{8}$$

which can be used to extend the zeta function to a meromorphic function on the complex plane with a simple pole of residue 1 at $s = 1$. It is important to notice that even though the series definition of the zeta function is not valid outside $\Re(s) > 1$, the functional equation allows us to deduce properties of $\zeta(s)$ for $\Re(s) < 0$ from those of the series. This leads to a fairly good understanding of the zeta function on these two half-planes save for the critical strip in between them. Inside that strip we cannot infer any properties of $\zeta(s)$ from those of the series. Hence other tools are needed, such as the approximate functional equation which we will introduce later.

Since $\zeta(s)$ is nonzero for $\Re(s) > 1$ the functional equation shows that the only zeros of $\zeta(s)$ on the left half-plane are the simple zeros at negative even integers arising from the poles of the Gamma function. These are called the trivial zeros of the zeta function and are well understood. Moreover, Riemann predicted that inside the critical strip $0 \leq \sigma \leq 1$ the zeta function has infinitely many zeros [17]—the nontrivial zeros. This was later strengthened to exclude the line $\Re(s) = 1$, and by symmetry $\Re(s) = 0$, independently by Hadamard [15, p. 50], and de la Vallée Poussin and Mertens [8, p. 106]. Let us denote the set of nontrivial zeros by Z, and an individual zero by $\rho = \beta + i\gamma$. When we talk about "zeros of the zeta function", we refer to these nontrivial zeros unless otherwise specified.

It is immediate from the series definition that $\overline{\zeta(\overline{s})} = \zeta(s)$ for $\Re(s) > 1$. By analytic continuation this relation must follow throughout the complex plane, so for $s \neq 1$ we have that $\overline{\zeta(s)} = \zeta(\overline{s})$. It follows that if $\rho \in Z$ then $\overline{\rho} \in Z$. More interestingly, from the functional equation we can see that if $\rho \in Z$ then $1-\rho \in Z$. Thus the nontrivial zeros of the zeta function exhibit symmetry about the line $\Re(s) = 1/2$. This is called the critical line. We are now in a position where we can state the Riemann Hypothesis rigorously[2].

Conjecture 1 (Riemann Hypothesis)

$$Z \subset \{s : \Re(s) = 1/2\}.$$

[2]If we are working under the Riemann Hypothesis, as is the case in this paper, we will denote the nontrivial zeros simply by $\rho = 1/2 + i\gamma$.

An important object of study is the number of zeros as a function of the height $T > 0$ along the imaginary axis. Let us denote this number by $N(T)$, that is

$$N(T) = |\{\rho \in Z : 0 < \Im(\rho) \leq T\}|.$$

Riemann gave an asymptotic formula for $N(T)$ without a proof. In 1895 von Mangoldt managed to prove his claim, namely, that

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T), \tag{9}$$

as $T \longrightarrow \infty$ [21, Theorem 9.4]. This is nowadays known as the Riemann-von Mangoldt Formula. Notice also that [15, §4.9]

$$N(T+1) - N(T) = O(\log T). \tag{10}$$

Albeit useful, the functional equation is not effective for computations since we have to work with infinite sums. Moreover, as we said above the series representation of the Riemann zeta function is not even valid inside the critical strip. To this end, Hardy and Littlewood developed the approximate functional equation in the early 1920s, which approximates the zeta function by a truncated sum and error terms [15, §1.9].

Theorem 2 (The Approximate Functional Equation for ζ) *Let h be a positive constant, then for $s \in \mathbb{C}$ with $0 < \sigma < 1$, $2\pi XY = |t|$, $X > h$, $Y > h$ we have*

$$\zeta(s) = \sum_{n<X} n^{-s} + \chi(s) \sum_{n \leq Y} n^{s-1} + O\Big(X^{-\sigma} \log|t|\Big) + O\Big(|t|^{\frac{1}{2}-\sigma} Y^{\sigma-1}\Big), \tag{11}$$

where $\chi(s) = \pi^{s-1/2}\Gamma\Big(\frac{(1-s)}{2}\Big)\Gamma\Big(\frac{s}{2}\Big)^{-1}$.

Notice that in the above equation both sums terminate sharply at X and Y, respectively. We say that this approximate functional equation has a sharp cutoff. It is also possible to introduce some weighting function to smooth out the tail of the sum. This yields a smooth cutoff. According to Rubinstein in [12, p. 443], the sharp cutoff works quite well in the case of the Riemann zeta function since the coefficient of each term is 1, whereas for Dirichlet L-functions smoothing has more of an effect.

3 Uniform Landau's Formula

3.1 Sums over Primes

It was Euler's discovery of his product formula in 1737 that revealed the first connection between primes and the Riemann zeta function. Since then, more and more results have surfaced tying the two intricately together. In such results it is common that the zeros of the zeta function play an important role. Riemann found, for example, an explicit formula for the amount of primes less than some number x. For our purposes it is more illuminating to consider a related formula for a function which we already saw in (2). There we found out that $\sum_{n<X} \Lambda(n)$ grows asymptotically as X, and that this summation is related to the Prime Number Theorem. In 1895 von Mangoldt proved the following explicit formula for the sum:

$$\sideset{}{'}\sum_{n\leq X} \Lambda(n) - X = -\sum_{\rho} \frac{X^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log\Big(1 - X^{-2}\Big),$$

where the prime in the summation means that if X is a power of prime then it is counted with a weight $1/2$. Here the second summation is taken over nontrivial zeros of the Riemann zeta function with the understanding that it is defined by the limit

$$\lim_{T\longrightarrow\infty} \sum_{|\gamma|<T} \frac{x^{\rho}}{\rho},$$

where γ is a height of a nontrivial zero [2, §7].

Summing over the zeros of the zeta function turns out to be a recurring theme when discussing prime numbers. Another example would be the so-called Landau's formula, which was discovered by the German mathematician Edmund Landau in 1911 in [10]. The formula states that for any fixed $x > 1$ we have

$$\sum_{0<\gamma\leq T} x^{\rho} = -\frac{T}{2\pi}\Lambda(x) + O(\log T)$$

as $T \longrightarrow \infty$. Here the sum runs over the positive imaginary parts of the Riemann zeros. We understand that $\Lambda(n)$ is extended to all of $\mathbb{R}$ by defining it to be 0 on $\mathbb{R} \setminus \mathbb{N}$. What is striking in this formula is that the right-hand side grows by a factor of T only if x is a prime power. So in a way the Riemann zeros single out the prime numbers from every other real number.

3.2 Gonek's Version of Landau's Formula

According to Gonek [5] the original version of Landau's Formula has a limited practical use since the estimate is not uniform in x. In his paper Gonek was able to prove a version of Landau's formula which is uniform in both x and T with only small sacrifices to the error term. Henceforth we shall refer to this result as the "Landau's formula".

Theorem 3 (Landau's Formula) *Let x, $T > 1$. Then*

$$\sum_{0<\gamma\leq T} x^{\rho} = -\frac{T}{2\pi}\Lambda(x) + O(x\log 2xT\ \log\log 3x) + O\left(\log x\ \min\left(T, \frac{x}{\langle x\rangle}\right)\right) + O\left(\log 2T\ \min\left(T, \frac{1}{\log x}\right)\right), \tag{12}$$

where $\langle x\rangle$ denotes the distance from x to the nearest prime power other than x itself.

We will first outline Gonek's proof of this result and then explain his application to prove a discrete mean value result for the Riemann zeta function. It is this application which will serve as an inspiration to our main result.

3.3 Proof

The main method in the proof is to construct a suitable rectangular contour integral and estimate it along each side separately. We will also need a technical lemma which we will state without proof as it does not add significant value into this paper.

Lemma 3 For x, $T > 1$ and $c = 1 + \frac{1}{\log 3x}$ we have

$$\sum_{\substack{n=2,\\ n\neq x}} \frac{\Lambda(n)}{n^c}\min\left(T, \frac{1}{|\log x/n|}\right) \ll \log 2x\ \log\log 3x + \log x\ \min\left(\frac{T}{x}, \frac{1}{\langle x\rangle}\right).$$

We suppose that T is not equal to the imaginary part of any nontrivial zero.

Let us define

$$\begin{aligned} I &= \frac{1}{2\pi i}\left(\int_{c+i}^{c+iT} + \int_{c+iT}^{1-c+iT} + \int_{1-c+iT}^{1-c+i} + \int_{1-c+i}^{c+i}\right)\frac{\zeta'(s)}{\zeta(s)}x^s\,\mathrm{d}s \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

It has been shown by direct computation that the first zero of the zeta function in the upper half-plane occurs when $\gamma > 14$. It follows from Cauchy's Residue Theorem that

$$I = \sum_{0<\gamma\le T} x^\rho.$$

Therefore it suffices to estimate each of the integrals.

Since $c > 1$, formula (7) on page 11 for the logarithmic derivative of the zeta function is valid so we can substitute it in. By uniform convergence we can exchange the order of integration and summation to get

$$\begin{aligned} I_1 &= -\sum_{n=2}^{\infty}\Lambda(n)(x/n)^c\left(\frac{1}{2\pi}\int_1^T (x/n)^{it}\,\mathrm{d}t\right) \\ &= -\frac{T-1}{2\pi}\Lambda(x) + O\left(\sum_{\substack{n=2\\ n\ne x}}^{\infty}\Lambda(n)\left(\frac{x}{n}\right)^c \min\left(T, \frac{1}{|\log {}^{x}\!/\!_{n}|}\right)\right), \end{aligned}$$

where we have separated the term with $n = x$. Also, the O-term estimate comes from taking the minimum of the trivial estimate $T - 1$ and the actual calculated integral. By Lemma 3 and the estimate $x^c \ll x$ we obtain the main term and the first two error terms in (12), save for the factor $\log T$ in the first estimate:

$$I_1 = -\frac{T}{2\pi}\Lambda(x) + O(x\log 2x\ \log\log 3x) + O\left(\log x\ \min\left(T, \frac{x}{\langle x\rangle}\right)\right).$$

This log-factor comes from I_2. For this term we estimate the logarithmic derivative of the zeta function with

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|\gamma-T|<1}\frac{1}{s-\rho} + O(\log 2T),$$

where $s = \sigma + iT$ [15, §3.3]. Substituting in gives us two integrals, one real and one complex, corresponding to the sum and the O-term respectively. We have

$$I_2 = \sum_{|\gamma - T| < 1} \int_{c+iT}^{1-c+iT} \frac{x^s}{s-\rho} \, \mathrm{d}s + O\left(\log 2T \int_{1-c}^{c} x^\sigma \, \mathrm{d}\sigma \right). \tag{13}$$

The integral in the error term is easily computed to be $(x^c - x^{1-c})/\log x$ so we have the following bound for the error,

$$\ll \log 2T \, \frac{x^c}{\log 2x} \ll x \frac{\log 2T}{\log 2x}.$$

To estimate the remaining sum we introduce a new contour of integration.

$$\int_{c+iT}^{1-c+iT} \frac{x^s}{s-\rho} \, \mathrm{d}s = -2\pi i x^\rho \delta(\rho) + \left(\int_{c+iT}^{c+i(T+1)} + \int_{c+i(T+1)}^{1-c+i(T+1)} + \int_{1-c+i(T+1)}^{1-c+iT} \right) \frac{x^s}{s-\rho} \, \mathrm{d}s$$

by Cauchy's Residue Theorem, where $\rho = \beta + i\gamma$ and $\delta(\rho) = 1$ if $T < \gamma < T+1$ and 0 otherwise. Notice that this integration is well-defined by our assumption that T is not equal to the imaginary part of a zero. Also note that the height $T+1$ is chosen simply because we wish to definitely move above the height of the zero in question. We can estimate the three integrals and the residue (since $0 < \beta < 1$) as

$$\ll x + x^c \int_T^{T+1} \frac{\mathrm{d}t}{|(c-\beta) + i(t-\gamma)|} + \int_{1-c}^{c} x^\sigma \, \mathrm{d}\sigma + \frac{x^{1-c}}{\beta - (1-c)}.$$

Here the estimate for the second integral is obtained by bounding the norm of the denominator from below by 1 since $|\gamma - T| < 1$. We work similarly for the last integral. Recall that $c = 1 + \frac{1}{\log 3x}$. Hence we can bound the above quantity to obtain

$$\ll x + x \int_\gamma^{\gamma+2} \min\left(\log 3x, \frac{1}{t-\gamma} \right) \mathrm{d}t + \frac{x}{\log 2x} + \log 3x. \tag{14}$$

In the last term in the above we have estimated x^{1-c} as $O(1)$ and the denominator as $|\beta - (1-c)| = |\beta + \log^{-1} 3x|$ which is trivially greater than $\log^{-1} 3x$. In the first integral we bound the integrand depending on whether the real part or the imaginary part has larger magnitude. We have also increased the length of the interval of integration. This is simply because the integrand is continuous and bounded (due to the min-function). It remains to estimate everything in (14) as $O(x \log\log 3x)$. This is trivial apart from the first integral. We need to separate this integral to two bits depending on which quantity in the min-function is smaller. First, notice that $\log 3x = (t-\gamma)^{-1}$ if and only if $t = \gamma + \log^{-1} 3x$. Since $(t-\gamma)^{-1}$

is decreasing we can separate the integral as follows:

$$\int_{\gamma}^{\gamma+2} \min\left(\log 3x, \frac{1}{t-\gamma}\right) \mathrm{d}t = \int_{\gamma}^{\gamma+\log^{-1} 3x} \log 3x \, \mathrm{d}t + \int_{\gamma+\log^{-1} 3x}^{\gamma+2} \frac{1}{t-\gamma} \, \mathrm{d}t.$$

The first integral is equal to precisely 1. The latter integral is equal to $\log 2 - \log\log^{-1} 3x$, which is $O(\log\log 3x)$ as required. Finally, we need to notice that by Riemann's estimate (10) of $N(T+1) - N(T)$, it follows that the summation in (13) has $O(\log 2T)$ terms. Therefore,

$$I_2 \ll x \log 2T \, \log\log 3x.$$

For I_3, we take a logarithmic derivative of the functional equation (8) and use the expansion (6) for the logarithmic derivative of the Gamma function to obtain

$$-\frac{\zeta'(1-c+it)}{\zeta(1-c+it)} = \frac{\zeta'(c-it)}{\zeta(c-it)} + \log \frac{t}{2\pi} + O\left(\frac{1}{t}\right)$$

for $t \geq 1$. Substituting this into I_3 gives us

$$I_3 = \frac{x^{1-c}}{2\pi} \int_1^T x^{it} \frac{\zeta'(c-it)}{\zeta(c-it)} \, \mathrm{d}t + \frac{x^{1-c}}{2\pi} \int_1^T x^{it} \log \frac{t}{2\pi} \, \mathrm{d}t + O(\log 2T).$$

Notice now that because of this trick we have managed to transfer ζ'/ζ to the domain where its series representation is valid. Substituting this in and integrating term-by-term yields

$$\begin{aligned} &\ll x^{1-c} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^c \log xn} \\ &\ll \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^c} = -\frac{\zeta'(c)}{\zeta(c)} \\ &\ll \frac{1}{c-1} = \log 3x \end{aligned}$$

for the first term since $\zeta'/\zeta(s)$ has a simple pole at $s = 1$, and all other poles are to the left of this point. By a trivial estimate, the second integral is $O(T \log 2T)$. On the other hand, by integration by parts it is $O(\log 2T / \log x)$. From this it follows that

$$I_3 \ll \log 2T + \log 3x + \min\left(T \log 2T, \frac{\log 2T}{\log x}\right).$$

Finally, I_4 is trivial to estimate since $\zeta'/\zeta(s)$ is bounded on $[1-c+i, c+i]$.

Hence the integral becomes

$$I_4 \ll \int_{1-c}^{c} x^{\sigma}\,\mathrm{d}\sigma \ll \frac{x}{\log 2x}.$$

Combining these estimates gives the required result in the case that T is not equal to imaginary part of any zero. If T happens to be equal to such a number then we need only change T by a bounded amount. Recalling our discussion of the function $N(T)$, it is not hard to see that a finite sum of the form $\sum_{T<\gamma\leq T+a} x^{\rho}$ contributes at most $O(x\log 2T)$, which is absorbed by the error terms in (12). This concludes the proof.

3.4 Gonek's Mean Value Estimate

As an application of this result Gonek proves the following theorem.

Theorem 4 *Assume the Riemann hypothesis. For large T and α real with $|\alpha| \leq \frac{1}{2\pi}\log T$, we have*

$$\sum_{0<\gamma\leq T} |\zeta(\tfrac{1}{2} + i(\gamma + 2\pi\alpha/\log T))|^2 = \left(1 - \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2\right)\frac{T}{2\pi}\log^2 T + O\Big(T\log^{7/4} T\Big),$$

where the constant implied by the O-term is absolute.

The main idea of the proof is to apply the approximate functional equation (11) and to rewrite the sums in the form that Landau's Formula can be applied. This will also be the method in our main theorem. Before that we need a few definitions.

4 Dirichlet L-functions

4.1 Basic Properties of χ

We will now put our discussion of the Riemann zeta function in to a new perspective by considering Dirichlet L-functions. These functions are one of many ways of generalising the usual Riemann zeta function. The basic idea is to introduce coefficients, which either vanish or are roots of unity, to each term in the series definition of $\zeta(s)$. These roots of unity are defined by characters on certain finite abelian groups.

A Dirichlet character χ modulo q is a group homomorphism $\chi\colon (\mathbb{Z}/q\mathbb{Z})^* \longrightarrow \mathbb{C}$. This Dirichlet character admits an extension to all integers by reducing the argument modulo q and defining $\chi(n) = 0$ for n with $(n,q) > 1$. Since χ is a group homomorphism it is clear that it defines a completely multiplicative arithmetic function. Moreover, if χ is a character then so is $\overline{\chi}$. We have, in fact, for $(n,q) = 1$ that

$$1 = |\chi(n)|^2 = \chi(n)\overline{\chi(n)}.$$

So we can define $\chi^{-1} = \overline{\chi}$. The principal character χ_0 is defined by

$$\chi_0(n) = \begin{cases} 1 & \text{if } (n,q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

χ_0 is also known as the trivial character. Furthermore, it is an example of a real character—a Dirichlet character assuming only real values. Non-real characters are called complex characters. There is one more characterising property for Dirichlet characters. We have, for complex or real characters, that $\chi(-1)^2 = \chi(1) = 1$ and thus

$$\chi(-1) = \pm 1.$$

According to whether this value is 1 or -1 we will call χ even or odd, respectively.

An important notion is that of the primitive character. First of all, we need to notice that as a function over integers our character has period q. It may turn out that this period is smaller than q, in which case we say that χ is imprimitive. Otherwise χ is primitive. Another way to understand this is to look at induced characters. Given $q' > 1$ such that $q' \mid q$ and a character χ' modulo q', we may define the induced character χ mod q by setting $\chi(n) = \chi'(n)$, for $(n,q) = 1$ (of course we then have $(n,q') = 1$). A character which does not arise as an induced character is primitive. Moreover, we call the least modulus (greater than 1) of a character its conductor.

The sum of the values of a character is easy to determine. Choose m such that

$\chi(m) \neq 0$, 1. We can do this if we assume that χ is non-principal. It follows that

$$\begin{aligned}\sum_{n=1}^{q} \chi(n) &= \sum_{n=1}^{q} \chi(nm)\\ &= \chi(m)\sum_{n=1}^{q} \chi(n)\\ &= 0, \end{aligned} \tag{15}$$

given that $\chi \neq \chi_0$. On the other hand, for a principal character modulo q we have trivially that

$$\sum_{n=1}^{q} \chi_0(n) = \varphi(q). \tag{16}$$

4.2 Basic Properties of $L(s,\chi)$

Now, given a Dirichlet character we can define the corresponding Dirichlet L-function.

Definition 6 A Dirichlet L-function for a Dirichlet character χ is defined for $\Re(s) > 1$ by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

This is clearly a uniformly convergent function by comparison with the Riemann zeta function. Thus it also defines an analytic function on $\Re(s) > 1$. Similarly, it has the Euler product

$$L(s,\chi) = \prod_p (1 - \chi(p)p^{-s})^{-1}.$$

From this is it follows that $L(s,\chi)$ is nonzero in the region of absolute convergence, as was the case with the zeta function.

All L-functions satisfy a certain functional equation as well, which allows extension to all of $\mathbb{C}$. However, we need to do a bit of work to get that far. One concept we will need is that of a Gauss sum. These are sums of the form

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) e^{\frac{2\pi i a}{q}}.$$

We have trivially that $|\tau(\chi)| \leq q$, but we can actually do better than this. It is true that for primitive characters $|\tau(\chi)| = \sqrt{q}$. We will show this in the case when the modulus of a non-principal χ is a prime, q. First, we start by evaluating the square to get that

$$|\tau(\chi)|^2 = \sum_{a=1}^{q-1}\sum_{b=1}^{q-1} \chi(a)\overline{\chi(b)}e^{\frac{2\pi i(a-b)}{q}},$$

since $\chi(q) = 0$. Reorder the inner sum to get n such that $b \equiv an \pmod{q}$. Then

$$\begin{aligned}|\tau(\chi)|^2 &= \sum_{a=1}^{q-q}\sum_{n=1}^{q-q} \chi(an^{-1}a^{-1})e^{\frac{2\pi i(a-an)}{q}} \\ &= (q-1)\overline{\chi(1)} + \sum_{n=2}^{q-1}\overline{\chi(n)}\sum_{a=1}^{q-1} e^{\frac{2\pi ia(1-n)}{q}}.\end{aligned}$$

where the inner sum is a sum of roots of unity with 1 removed (i.e. of the form $\omega + \omega^2 + \ldots + \omega^{q-1}$, where ω is a q-th root of unity) and so is equal to -1. Hence

$$|\tau(\chi)|^2 = q - \sum_{n=1}^{q-1}\overline{\chi(n)},$$

which is equal to q by (15). For the general case see Davenport [2, pp. 66-67].

The functional equation is valid only for primitive characters. Moreover, it is different for even and odd characters, and requires a slightly different proof in each case [2, §9]. The main idea is to start with the definition of the Gamma function and manipulate it into the desired form. Eventually we arrive at the following:

$$L(s,\chi) = \frac{\tau(\chi)i^{-\mathfrak{a}}}{\sqrt{q}}\left(\frac{q}{\pi}\right)^{\frac{1}{2}(2s-1)}\frac{\Gamma\left(\frac{1-s+\mathfrak{a}}{2}\right)}{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)}L(1-s,\overline{\chi}), \tag{17}$$

where the symbol $\mathfrak{a}$ is defined by

$$\mathfrak{a} = \frac{1-\chi(-1)}{2}.$$

Notice that, unlike with the zeta function, this time the functional equation does not tell us nearly as much information about the zeros of the corresponding function unless, of course, the character is real. What we do know, though, is that since the series representation is absolutely convergent in the region $\Re(s) > 1$, $L(s,\chi)$ is nonzero there. Thus, by the functional equation it follows that $L(s,\chi)$, for a primitive character χ, has trivial zeros only at negative even integers, just like the Riemann zeta function. On the other hand, it is possible to show that $L(1,\chi)$

is finite for non-principal characters and, moreover, that the series defining it is convergent conditionally. This is in stark contrast to the pole of the Riemann zeta function. It is actually possible to deduce Dirichlet's Theorem for primes in arithmetic progressions from the fact that $L(1,\chi) \neq 0$ for non-principal characters (see Murty [14, § 2]). It follows that $L(s,\chi)$ has a trivial zero at $s = 0$ as well, whenever χ is non-principal.

4.3 An Approximate Functional Equation for $L(s,\chi)$

It makes sense to expect that Dirichlet L-functions should satisfy some kind of an approximate functional equation obtained from (17). There are actually many forms of this, but the one we will be using is given by Lavrik [11].

Theorem 5 (The Approximate Functional Equation for L) *Let χ be a primitive character mod q. Then, for $s = \sigma + it$ with $0 < \sigma < 1$, $t > 0$, and $x = \Delta\sqrt{\frac{qt}{2\pi}}$, $y = \Delta^{-1}\sqrt{\frac{qt}{2\pi}}$, and $\Delta \geq 1$, $\Delta \in \mathbb{N}$, we have*

$$L(s,\chi) = \sum_{n \leq x} \frac{\chi(n)}{n^s} + \varepsilon(\chi)\left(\frac{q}{\pi}\right)^{\frac{1}{2}-s} \frac{\Gamma\left(\frac{1-s+\mathfrak{a}}{2}\right)}{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)} \sum_{n \leq y} \frac{\overline{\chi(n)}}{n^{1-s}} + R_{xy}, \tag{18}$$

where $\varepsilon(\chi) = q^{-1/2} i^{\mathfrak{a}} \tau(\chi)$ and $\tau(\chi) = \sum_{a=1}^{q} \chi(a) e^{\frac{2\pi i a}{q}}$. Moreover,

$$R_{xy} \ll \sqrt{q}\left(y^{-\sigma} + x^{\sigma-1}(qt)^{1/2-\sigma}\right) \log 2t,$$

and in particular, for $x = y$,

$$R \ll x^{-\sigma}\sqrt{q} \log 2t.$$

On the critical line we have for $q \leq t(\log^4 2t)^{-1}$

$$R \ll \left(\frac{q}{t}\right)^{1/4} \log 2t \ll 1.$$

Notice that we are still working with a sharp cutoff. This is because it is sufficient for our needs, and a sharp cutoff was also employed by Gonek and Petridis. It might be possible to extend our result to the critical line by switching to a smooth cutoff. An example of such a formula can be found in [12, pp. 443–445].

In order to be applicable to our proof we need a stronger version of the above formula, where we allow Δ to be a positive real number greater than 1.

Lemma 4 The approximate functional equation (18) is valid for $\Delta \in \mathbb{R}_{\geq 1}$ with the additional error terms $O\left(t^{-\sigma/2}\right)$ and $O\left(t^{(1-3\sigma)/2}\right)$ provided that χ is non-principal.

Proof. Let $c = \sqrt{qt/2\pi}$, and suppose $\Delta \in \mathbb{R}_{\geq 1}$. Without loss, we can also suppose that Δ is not an integer. We will denote the integral part of Δ by $[\Delta]$. Then

$$\begin{aligned} L(s,\chi) &= \sum_{n \leq c\Delta} \frac{\chi(n)}{n^s} + \varepsilon(\chi)\left(\frac{q}{\pi}\right)^{\frac{1}{2}-s} \frac{\Gamma\left(\frac{1-s+\mathfrak{a}}{2}\right)}{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)} \sum_{n \leq c\Delta^{-1}} \frac{\overline{\chi(n)}}{n^{1-s}} + R_{xy} \\ &\quad - \sum_{c[\Delta] < n \leq c\Delta} \frac{\chi(n)}{n^s} + \varepsilon(\chi)\left(\frac{q}{\pi}\right)^{\frac{1}{2}-s} \frac{\Gamma\left(\frac{1-s+\mathfrak{a}}{2}\right)}{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)} \sum_{c\Delta^{-1} < n \leq c[\Delta]^{-1}} \frac{\overline{\chi(n)}}{n^{1-s}}. \end{aligned}$$

So we need to estimate the last two sums. By (33) we find that the factor in front of the second sum is $O\left(t^{1-2\sigma}\right)$. Denote $X(x) = \sum_{n \leq x} \chi(n)$ so that by (15) we have $X(x) = O(1)$. Now, applying summation by parts gives

$$\begin{aligned} \sum_{c[\Delta] < n \leq c\Delta} \frac{\chi(n)}{n^s} &= (c\Delta)^{-s} X(c\Delta) - (c[\Delta])^{-s} X(c[\Delta]) - \int_{c[\Delta]}^{c\Delta} X(x) \frac{\mathrm{d}x^{-s}}{\mathrm{d}x}\,\mathrm{d}x \\ &\ll c^{-\sigma} + \left[x^{-\sigma}\right]_{c[\Delta]}^{c\Delta} \\ &\ll t^{-\sigma/2}. \end{aligned}$$

Similarly for the other sum we obtain

$$\sum_{c\Delta^{-1} < n \leq c[\Delta]^{-1}} \frac{\overline{\chi(n)}}{n^{1-s}} \ll t^{(\sigma-1)/2}.$$

Combining these three estimates gives

$$\begin{aligned} L(s,\chi) &= \sum_{n \leq c\Delta} \frac{\chi(n)}{n^s} + \varepsilon(\chi)\left(\frac{q}{\pi}\right)^{\frac{1}{2}-s} \frac{\Gamma\left(\frac{1-s+\mathfrak{a}}{2}\right)}{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)} \sum_{n \leq c\Delta} \frac{\overline{\chi(n)}}{n^{1-s}} + R_{xy} \\ &\quad + O\left(t^{-\sigma/2}\right) + O\left(t^{\frac{1-3\sigma}{2}}\right). \end{aligned}$$

It is useful to notice that in the case $\sigma \in (\frac{1}{2}, 1)$ the final two error terms are $O\left(t^{-1/4}\right)$. □

5 Away from the Critical Line

5.1 Main Theorem

In this section we will state and prove our main theorem. The proof will follow in the steps of [5, Theorem 2] and [16, Theorem 1.9]. Their proofs are for the Riemann zeta function and GL_2 L-functions, respectively. In this section s will denote $\sigma + it$ or $\sigma + i\gamma$, depending on the context, where $\sigma \in (\frac{5}{6}, 1)$.

Main Theorem *Assume the Riemann Hypothesis. Let χ_1, χ_2 be real, primitive, and non-principal characters mod q and l, respectively, where $q \neq l$ are primes. Then, for a positive proportion of the nontrivial zeros of $\zeta(s)$, for a fixed $\sigma \in (\frac{5}{6}, 1)$, the two Dirichlet L-functions $L(\sigma + i\gamma, \chi_1)$ and $L(\sigma + i\gamma, \chi_2)$ are linearly independent over $\mathbb{R}$.*

Two nonzero complex numbers z and w being linearly independent over the reals means that there is no $a \in \mathbb{R}$ such that $z = aw$. This is equivalent to the quotient z/w being non-real. By considering the imaginary part this is then equivalent to $|z\overline{w} - \overline{z}w| > 0$. In this theorem we are considering the case when z and w are values of Dirichlet L-functions. Instead of looking at these functions at a single point, we will average over multiple points with a fixed real part $\sigma \in (\frac{5}{6}, 1)$ and the imaginary part at the height of the Riemann zeros. So, in essence, we are considering the discrete mean value of a product of two L-functions.

In this paper we are assuming the RH purely because it makes the proof simpler. This is because expressions of the form x^ρ become easier to deal with if we know the real part explicitly. On the other hand, the distribution of these specific points does not seem to have any impact on the rest of the proof. Hence it is reasonable to suspect that the RH is not an essential requirement. In fact, following Gonek [6], it might be possible to obtain the result without the RH by integrating

$$\frac{\zeta'(s)}{\zeta(s)} B(s, P) L(s, \chi_1) \overline{L(s, \chi_2)}$$

over a suitable contour. This would pick the desired points as residues of the integrand yielding the required sum. This idea is also used in the proof of Landau's formula in section 3.

The requirement for a primitive character comes simply from the fact that our approximate functional equation, Theorem 5, has that requirement. Approximate functional equations for imprimitive characters do exists, but they are more complicated. Also in our proof we need to estimate sums of the form

$$\sum_{m=1}^{T} \chi(m).$$

If χ is non-principal then we saw that the sum over one period is 0, so we can bound the above sum independently of T. On the other hand if χ is principal then the sum over a period would be $\varphi(q)$, where q is the conductor of χ. Hence, allowing principal characters would result in unwanted growth in certain terms.

As a final remark about our assumptions, it should be noted that the requirement that q and l are primes is again trivial. We need these two numbers to have some coprime factors in order for Proposition 3 to work, whereas this has no impact on the other two propositions. However, for definiteness we have chosen to just work with primes.

The proof will be divided into three propositions after which the main result follows easily. In the first proposition we want to calculate discrete mean values of sums of terms of the type $z\overline{w}$ and $\overline{z}w$. Now, if we subtract one of these mean values from the other, we get a sum of terms of the form $z\overline{w} - \overline{z}w$. Each term is nonzero precisely when the two numbers are linearly independent over reals. Hence we need to prove that the two mean values are not equal, which is the content of Proposition 3. Finally, we get the main result by applying the Cauchy-Schwarz inequality to a sum of such terms. In doing so, we find that we need to estimate a sum of squares of the absolute values of the above quantities, that is, $|z\overline{w} - \overline{z}w|^2$. This is done in Proposition 2.

The first problem in the proof of our result is that the mean values of $z\overline{w}$ and $\overline{z}w$ will be the same, because the characters are real. Thus we need to introduce some kind of weighting in order to shove these sums off balance. We do this by introducing a finite Dirichlet polynomial, B, which cancels some terms from either of the L-functions, depending on which mean value we are considering. So let us define

$$B(s, P) = \prod_{p \le P} (1 - \chi_1(p)p^{-s})(1 - \chi_2(p)p^{-s}) \tag{19}$$

for some fixed prime P to be determined in Proposition 3. Let us also assume that this polynomial has the Dirichlet series $B(s,P) = \sum_{n\le R} c_n n^{-s}$, for some R depending on P. It is useful to notice that if we expand the product in the definition of B then the coefficient of any c_p is bounded above in absolute value by 2. Hence, as there are at most P primes, we have for all n

$$|c_n| \le 2^P. \tag{20}$$

With this weight introduced we find the mean values to be the following.

Proposition 1

$$\sum_{0<\gamma\le T} B(s,P)L(s,\chi_1)\overline{L(s,\chi_2)} \sim N(T)\sum_{n=1}^{\infty}\frac{d_n\chi_2(n)}{n^{2\sigma}}, \tag{21}$$

and

$$\sum_{0<\gamma\le T} B(s,P)\overline{L(s,\chi_1)}L(s,\chi_2) \sim N(T)\sum_{n=1}^{\infty}\frac{e_n\chi_1(n)}{n^{2\sigma}}, \tag{22}$$

where

$$B(s,P)L(s,\chi_1) = \sum_{n=1}^{\infty}\frac{d_n}{n^s}, \qquad B(s,P)L(s,\chi_2) = \sum_{n=1}^{\infty}\frac{e_n}{n^s}.$$

It is worthwhile to remark that if we were working with complex characters instead, then the characters that appear in the mean values would be complex conjugates. The upshot of this is that even without the weighting the mean values would still not be equal. This is why we cannot apply the same approach, which is currently used in Proposition 3, to deal with the complex case.

Proposition 2 *Let*

$$A(\gamma) = B(s,P)\Big(L(s,\chi_1)\overline{L(s,\chi_2)} - \overline{L(s,\chi_1)}L(s,\chi_2)\Big).$$

Then, under the Riemann Hypothesis,

$$\sum_{0<\gamma\le T} |A(\gamma)|^2 \ll N(T). \tag{23}$$

Proposition 3 *Under the Riemann Hypothesis*

$$\sum_{\gamma\le T} A(\gamma) \sim C\cdot N(T) \tag{24}$$

for some nonzero constant C.

As said above, the requirement for a real character comes from this proposition. In order to show that the C is nonzero we consider the Euler factors of both Dirichlet series. For real characters we find that most of them are the same, which allows us to consider only a finite number of factors at a time. On the other hand, the first two propositions work fine with complex characters. The third proposition would reduce to showing that $L(2\sigma, \chi_1\overline{\chi_2})$ is not real. The author is not aware of any results on this.

With the three propositions proved, the main theorem follows immediately. By the Cauchy-Schwarz inequality and Propositions 2 and 3

$$\sum_{\substack{0<\gamma\leq T\\ A(\gamma)\neq 0}} 1 \geq \frac{|\sum_{0<\gamma\leq T} A(\gamma)|^2}{\sum_{0<\gamma\leq T}|A(\gamma)|^2} \gg \frac{|C|^2 N(T)^2}{N(T)} = |C|^2 N(T).$$

This proves that a positive proportion of the $A(\gamma)$'s are nonzero; in particular, for the same γ's, $L(s, \chi_1)$ and $L(s, \chi_2)$ are linearly independent over the reals. □

5.2 Proposition 1

It is clear that $d_n \ll 1$ as the d_n's contain only products of characters[3] and, in particular, define a multiplicative arithmetic function. Let us now define, for a fixed t,

$$B(s, P) \sum_{n\leq\sqrt{\frac{qlt}{2\pi}}} \chi_1(n)n^{-s} = \sum_{n\leq R\sqrt{\frac{qlt}{2\pi}}} \frac{d'_n}{n^s}.$$

So the d'_n coefficients are similar to those in Dirichlet convolutions. Recall first that c_n are defined by expanding $B(s, P)$ into a sum, so $|c_n| \leq 2^P$. Moreover, it is illuminating to notice for future purposes that many of the c_i's are in fact equal to 0. We have

$$d_n = \sum_{n=km} c_k\chi_1(m), \tag{25}$$

and hence for $n \leq R\sqrt{\frac{qlt}{2\pi}}$

$$d'_n = \sum_{\substack{n=km\\ k\leq R}} c_k\chi_1(m). \tag{26}$$

[3]See Proposition 3 for an explicit calculation.

From this it follows that $d_n = d'_n$ for $n \leq \sqrt{\frac{qlt}{2\pi}}$. We also need to show that $d'_n \ll 1$.

Let $p_1, \dots, p_h$, for some $h > 1$, denote all of the primes below P in an increasing order. Define $\tilde{P} = p_1 \dots p_h P$. From the product representation (19) of $B(s, P)$, we can see that $c_n = 0$ for $n > 1$ if n contains any prime factors greater than P. This happens, in particular, if $(n, \tilde{P}) = 1$. Hence for such n, $d'_n = \chi_1(n)$. On the other hand, all the primes in $B(s, P)$ occur with multiplicity of at most two. So $c_{P^3} = 0$, for example. Thus we only need to consider $n > 1$ with $n = p_1^{\alpha_1} \dots p_h^{\alpha_h} P^{\alpha_0}$, where $0 \leq \alpha_i \leq 2$ for all i, and not all alphas are zero. It follows that any d'_n can have at most $(h+1)^3$ summands. Thus, recalling our estimate (20) on c_n, we find that $|d'_n| \leq 2^P (h+1)^3$. In particular, $d'_n \ll 1$ as required.

The approximate functional equation, with our Lemma 4, for χ_1 with $\Delta = \sqrt{l}$ gives

$$L(s, \chi_1) = \sum_{n \leq \sqrt{\frac{qlt}{2\pi}}} \chi_1(n) n^{-s} + \xi(s, \chi_1) \sum_{n \leq \sqrt{\frac{qt}{2\pi l}}} \chi_1(n) n^{s-1} + O\Big(t^{-\sigma/2} \log t\Big) + O\Big(t^{-1/4}\Big),$$

where

$$\xi(s, \chi) = \varepsilon(\chi) \Big(\frac{q}{\pi}\Big)^{1/2-s} \frac{\Gamma\Big(\frac{1-s+\mathfrak{a}}{2}\Big)}{\Gamma\Big(\frac{s+\mathfrak{a}}{2}\Big)}.$$

Similarly for χ_2 with $\Delta = \sqrt{q}R$ we get

$$\begin{aligned} L(s, \chi_2) = \sum_{n \leq R\sqrt{\frac{qlt}{2\pi}}} \chi_2(n) n^{-s} + \xi(s, \chi_2) \sum_{n \leq \frac{1}{R}\sqrt{\frac{lt}{2\pi q}}} \chi_2(n) n^{s-1} \\ + O\Big(t^{-\sigma/2} \log t\Big) + O\Big(t^{-1/4}\Big). \end{aligned}$$

We can now expand the left-hand side in (21) to

$$\begin{aligned} \sum_{0<\gamma\leq T} B(s, P) \Bigg(\sum_{n \leq \sqrt{\frac{ql\gamma}{2\pi}}} \chi_1(n) n^{-s} + \xi(s, \chi_1) \sum_{n \leq \sqrt{\frac{q\gamma}{2\pi l}}} \chi_1(n) n^{s-1} + O\Big(\gamma^{-\sigma/2} \log \gamma\Big) \\ + O\Big(\gamma^{-1/4}\Big) \Bigg) \times \Bigg(\sum_{n \leq R\sqrt{\frac{ql\gamma}{2\pi}}} \chi_2(n) n^{-\bar{s}} + \overline{\xi(s, \chi_2)} \sum_{n \leq \frac{1}{R}\sqrt{\frac{l\gamma}{2\pi q}}} \chi_2(n) n^{\bar{s}-1} \\ + O\Big(\gamma^{-\sigma/2} \log \gamma\Big) + O\Big(\gamma^{-1/4}\Big) \Bigg). \quad (27) \end{aligned}$$

The main term comes from the diagonal entries

$$\sum_{0<\gamma\le T} B(s,P) \sum_{n\le\sqrt{\frac{ql\gamma}{2\pi}}} \chi_1(n) n^{-s} \sum_{n\le R\sqrt{\frac{ql\gamma}{2\pi}}} \chi_2(n) n^{-\bar{s}},$$

which is equal to

$$\begin{aligned}
&\sum_{0<\gamma\le T}\ \sum_{n\le R\sqrt{\frac{ql\gamma}{2\pi}}} d'_n n^{-\sigma-i\gamma} \sum_{n\le R\sqrt{\frac{ql\gamma}{2\pi}}} \chi_2(n) n^{-\sigma+i\gamma} \\
&= \sum_{0<\gamma\le T}\left(\sum_{n\le R\sqrt{\frac{ql\gamma}{2\pi}}} \frac{d'_n\chi_2(n)}{n^{2\sigma}} + \sum_{n\ne m}^{R(ql\gamma/2\pi)^{1/2}} \frac{d'_m\chi_2(n)}{(nm)^{\sigma}}\left(\frac{n}{m}\right)^{i\gamma}\right) \\
&= Z_1 + Z_2. \qquad (28)
\end{aligned}$$

The required term in Equation (21) is given by Z_1.

$$\begin{aligned}
Z_1 &= \sum_{0<\gamma\le T}\left(\sum_{n=1}^{\infty}\frac{d_n\chi_2(n)}{n^{2\sigma}} - \sum_{n>R\sqrt{\frac{ql\gamma}{2\pi}}}\frac{d_n\chi_2(n)}{n^{2\sigma}} + \sum_{n\le R\sqrt{\frac{ql\gamma}{2\pi}}}\frac{(d'_n-d_n)\chi_2(n)}{n^{2\sigma}}\right) \\
&= N(T)\sum_{n=1}^{\infty}\frac{d_n\chi_2(n)}{n^{2\sigma}} + C_1 + C_2,
\end{aligned}$$

where the series converges by our asymptotics for d_n and the fact that $\sigma > 1/2$. We need to estimate C_1 and C_2. For C_1 we have

$$\begin{aligned}
C_1 &\ll \sum_{0<\gamma\le T}\ \sum_{n>\gamma^{1/2}} n^{-2\sigma} \\
&\ll \sum_{0<\gamma\le T}\left(\int_{\gamma^{1/2}}^{\infty} x^{-2\sigma}\,\mathrm{d}x + \gamma^{-\sigma}\right) \\
&\ll \sum_{0<\gamma\le T}\gamma^{1/2-\sigma} \ll T^{1/2-\sigma}N(T) = o(N(T)).
\end{aligned}$$

Similarly,

$$\begin{aligned}
C_2 &\ll \sum_{0<\gamma\le T}\ \sum_{n>\sqrt{\frac{ql\gamma}{2\pi}}} n^{-2\sigma} \\
&\ll \sum_{0<\gamma\le T}\ \sum_{n>\gamma^{1/2}} n^{-2\sigma} = o(N(T)).
\end{aligned}$$

To estimate Z_2 we wish to exchange the order of summation and apply Landau's formula (12). Splitting and rewriting Z_2 in terms of the zeros of $\zeta(s)$ we get

$$\begin{aligned} Z_2 &= \sum_{0<\gamma\leq T} \sum_{n\leq R\sqrt{\frac{ql\gamma}{2\pi}}} \sum_{m<n} \left(\frac{d'_m \chi_2(n)}{n^{\sigma+1/2} m^{\sigma-1/2}} \left(\frac{n}{m}\right)^{1/2+i\gamma} + \frac{d'_n \chi_2(m)}{n^{\sigma+1/2} m^{\sigma-1/2}} \overline{\left(\frac{n}{m}\right)^{1/2+i\gamma}} \right) \\ &= \sum_{n\leq R\sqrt{\frac{qlT}{2\pi}}} \sum_{m<n} \sum_{\frac{2\pi n^2}{qlR^2}\leq\gamma\leq T} \left(\frac{d'_m \chi_2(n)}{n^{\sigma+1/2} m^{\sigma-1/2}} \left(\frac{n}{m}\right)^{\rho} + \frac{d'_n \chi_2(m)}{n^{\sigma+1/2} m^{\sigma-1/2}} \overline{\left(\frac{n}{m}\right)^{\rho}} \right). \end{aligned}$$

To apply Landau's formula we split the innermost sum to $0 < \gamma \leq T$ and $0 < \gamma \leq 2\pi n^2/qlR^2$. Hence,

$$\begin{aligned} Z_2 = &-\frac{T}{2\pi} \sum_{n\leq R\sqrt{\frac{qlT}{2\pi}}} \sum_{m<n} \frac{d'_m \chi_2(n) + d'_n \chi_2(m)}{n^{\sigma+1/2} m^{\sigma-1/2}} \Lambda\left(\frac{n}{m}\right) \\ &+ O\left(\sum_{n\leq R\sqrt{\frac{qlT}{2\pi}}} \sum_{m<n} \frac{n^2 \Lambda(\frac{n}{m})}{n^{\sigma+1/2} m^{\sigma-1/2}} \right) \\ &+ O\left(\sum_{n\leq R\sqrt{\frac{qlT}{2\pi}}} \sum_{m<n} \frac{1}{n^{\sigma+1/2} m^{\sigma-1/2}} \frac{n}{m} \log\frac{2nT}{m} \log\log\frac{3n}{m} \right) \\ &+ O\left(\sum_{n\leq R\sqrt{\frac{qlT}{2\pi}}} \sum_{m<n} \frac{1}{n^{\sigma+1/2} m^{\sigma-1/2}} \log\frac{n}{m} \min\left(T, \frac{n/m}{\langle n/m \rangle}\right) \right) \\ &+ O\left(\log T \sum_{n\leq R\sqrt{\frac{qlT}{2\pi}}} \sum_{m<n} \frac{1}{n^{\sigma+1/2} m^{\sigma-1/2}} \min\left(T, \frac{1}{\log(n/m)}\right) \right) \\ &= Z_{21} + Z_{22} + Z_{23} + Z_{24} + Z_{25}. \end{aligned}$$

We begin by estimating Z_{21}. The term in m & n vanishes if $m \nmid n$. Thus we can write $n = km$ and obtain

$$\begin{aligned} Z_{21} &\ll \frac{T}{2\pi} \sum_{k\leq R\sqrt{\frac{qlT}{2\pi}}} \sum_{m<\frac{R}{k}\sqrt{\frac{qlT}{2\pi}}} \frac{\Lambda(k)}{k^{\sigma+1/2} m^{2\sigma}} \\ &\ll \frac{T}{2\pi} \sum_{k\leq R\sqrt{\frac{qlT}{2\pi}}} k^{\epsilon-\sigma-1/2} \sum_{m<\frac{R}{k}\sqrt{\frac{qlT}{2\pi}}} m^{-2\sigma} \end{aligned}$$

since $\Lambda(k) \ll k^{\epsilon}$ for any $\epsilon > 0$. Since both sums are partial sums of convergent series we get

$$Z_{21} = O(T).$$

Working similarly with Z_{22} gives

$$\begin{aligned}
Z_{22} &\ll \sum_{k\le R\sqrt{\frac{qlT}{2\pi}}} \sum_{m<\frac{R}{k}\sqrt{\frac{qlT}{2\pi}}} \frac{\Lambda(k)}{k^{\sigma-3/2}m^{2\sigma-2}} \\
&\ll \sum_{k\le R\sqrt{\frac{qlT}{2\pi}}} k^{3/2-\sigma+\epsilon} \sum_{m<\frac{R}{k}\sqrt{\frac{qlT}{2\pi}}} m^{2-2\sigma} \\
&\ll \sum_{k\le R\sqrt{\frac{qlT}{2\pi}}} k^{3/2-\sigma+\epsilon} \left(\int_1^{\frac{R}{k}\sqrt{\frac{qlT}{2\pi}}} x^{2-2\sigma}\,\mathrm{d}x + 1 \right) \\
&\ll \sum_{k\le R\sqrt{\frac{qlT}{2\pi}}} k^{3/2-\sigma+\epsilon} \left(\left(\frac{T^{1/2}}{k} \right)^{3-2\sigma} + 1 \right) \\
&\ll T^{\frac{3-2\sigma}{2}} \sum_{k\le R\sqrt{\frac{qlT}{2\pi}}} k^{3/2-\sigma+\epsilon+2\sigma-3} \\
&\ll T^{\frac{3-2\sigma}{2}} \sum_{k\ll T^{1/2}} k^{\sigma-3/2+\epsilon} \\
&\ll T^{\frac{3-2\sigma}{2}} T^{\frac{\sigma-1/2+\epsilon}{2}} = T^{\frac{5/2-\sigma+\epsilon}{2}} = O(T).
\end{aligned}$$

For Z_{23} we do not get a divisibility condition, but we can just look at the sums as is.

$$\begin{aligned}
Z_{23} &\ll \log T\ \log\log T \sum_{n\le R\sqrt{\frac{qlT}{2\pi}}} \frac{1}{n^{\sigma-1/2}} \sum_{m<n} \frac{1}{m^{\sigma+1/2}} \\
&\ll \log T\ \log\log T \sum_{n\le R\sqrt{\frac{qlT}{2\pi}}} \frac{1}{n^{\sigma-1/2}} \\
&\ll T^{\frac{3/2-\sigma}{2}} \log T\ \log\log T = o(N(T)),
\end{aligned}$$

because the sum over m is a partial sum of a convergent series.

In order to estimate Z_{24} we write $n = lm + r$, where $-\frac{m}{2} < r \le \frac{m}{2}$. Hence

$$\left\langle l + \frac{r}{m} \right\rangle = \begin{cases} \frac{|r|}{m}, & \text{if } l \text{ is a prime power and } r \ne 0, \\ \ge \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Let $c = R\sqrt{\frac{ql}{2\pi}}$ then $\frac{n}{m} \le n \le c\sqrt{T}$, and so

$$\begin{aligned}
Z_{24} &\ll \log T \sum_{n \le cT^{1/2}} \sum_{m<n} \frac{1}{n^{\sigma+1/2}m^{\sigma-1/2}} \frac{n}{m} \frac{1}{\langle n/m \rangle} \\
&\ll \log T \sum_{m \le cT^{1/2}} \sum_{l \le \lfloor cT^{1/2}/m \rfloor + 1} \sum_{-m/2 < r \le m/2} \frac{1}{m^{\sigma+1/2}(lm+r)^{\sigma-1/2}} \frac{1}{\langle l + \frac{r}{m} \rangle} \\
&\ll \log T \sum_{m \le cT^{1/2}} \sum_{l \le \lfloor cT^{1/2}/m \rfloor + 1} \left(\Lambda(l) \frac{m \log m}{m^{\sigma+1/2}(lm)^{\sigma-1/2}} + \frac{m}{m^{\sigma+1/2}(lm)^{\sigma-1/2}} \right) \\
&\ll \log T \sum_{m \le cT^{1/2}} \frac{\log m}{m^{2\sigma-1}} \sum_{l \ll cT^{1/2}/m} \frac{l^{\epsilon}}{l^{\sigma-1/2}} \\
&\ll T^{\frac{3/2-\sigma+\epsilon}{2}} \log T \sum_{m \le cT^{1/2}} \frac{\log m}{m^{\sigma+1/2+\epsilon}} = O(T),
\end{aligned}$$

since the last sum is a partial sum of a convergent series.

Finally, for Z_{25} set $m = n - r$, $1 \le r \le n-1$. So in particular

$$\log \frac{n}{m} > -\log(1 - \frac{r}{n}) > \frac{r}{n}.$$

Hence,

$$\begin{aligned}
Z_{25} &\ll \log T \sum_{n \le cT^{1/2}} \sum_{1 \le r < n} \frac{1}{n^{\sigma+1/2}(n-r)^{\sigma-1/2}} \frac{n}{r} \\
&\ll \log T \sum_{n \le cT^{1/2}} \frac{1}{n^{\sigma-1/2}} \sum_{r \le n-1} \frac{1}{r} \\
&\ll \log T \sum_{n \le cT^{1/2}} \frac{\log n + O(1)}{n^{\sigma-1/2}} \\
&\ll \log T \int_1^{cT^{1/2}} x^{1/2-\sigma+\epsilon} \, \mathrm{d}x \\
&\ll T^{\frac{3/2-\sigma+\epsilon}{2}} \log T = O(T).
\end{aligned}$$

It remains to estimate all the other terms in (27). By repeating the analysis done for Z_1 and Z_2 but with first d'_n replacing $\chi_2(n)$, and then the other way around, we obtain the following estimates

$$\sum_{\gamma \le T} \left| \sum_{n \le R\sqrt{\frac{ql\gamma}{2\pi}}} d'_n n^{-\sigma-i\gamma} \right|^2 \ll N(T), \tag{29}$$

$$\sum_{\gamma\le T}\left|\sum_{n\le R\sqrt{\frac{ql\gamma}{2\pi}}}\chi_2(n)n^{-\sigma+i\gamma}\right|^2 \ll N(T). \tag{30}$$

With trivial changes to the above argument we get,

$$\sum_{\gamma\le T}\left|\sum_{n\le\sqrt{\frac{q\gamma}{2\pi l}}}\chi_1(n)n^{\sigma-1+i\gamma}\right|^2 \ll T^{\sigma-1/2}N(T), \tag{31}$$

$$\sum_{\gamma\le T}\left|\sum_{n\le\frac{1}{R}\sqrt{\frac{l\gamma}{2\pi q}}}\chi_2(n)n^{\sigma-1-i\gamma}\right|^2 \ll T^{\sigma-1/2}N(T). \tag{32}$$

For example, if we denote the sum in (31) by S then

$$\begin{aligned} S &= \sum_{\gamma\le T}\left(\sum_{n\le\sqrt{\frac{q\gamma}{2\pi l}}}\chi_1(n^2)n^{2\sigma-2} + \sum_{m\ne n}^{\sqrt{\frac{q\gamma}{2\pi l}}}\frac{\chi_1(n)\chi_1(m)}{(nm)^{1-\sigma}}\left(\frac{n}{m}\right)^{i\gamma}\right)\\ &= S_1 + S_2. \end{aligned}$$

Now,

$$\begin{aligned} S_1 &\ll \sum_{\gamma\le T}\gamma^{\frac{2\sigma-1}{2}}\\ &\ll T^{\sigma-1/2}N(T), \end{aligned}$$

and we split S_2 as we did with Z_2 and proceed in the same way.

We also need to estimate the order of growth of the derivative of $|\xi(s,\chi)|^2$. Recall that for a primitive χ modulo a prime

$$|\tau(\chi)| = q^{1/2}.$$

First, notice that $|\varepsilon(\chi)| = 1$ so

$$|\xi(s,\chi)| = \left(\frac{q}{\pi}\right)^{1/2-\sigma}\left|\Gamma\left(\frac{1-s+\mathfrak{a}}{2}\right)\right|\left|\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)\right|^{-1},$$

and by the Asymptotic Stirling's Formula (5),

$$|\xi(s,\chi)|^2 \sim A\left(\frac{q}{\pi}\right)^{1-2\sigma}\gamma^{1-2\sigma}, \tag{33}$$

as $\overline{\Gamma(z)} = \Gamma(\overline{z})$, where A is some nonzero constant. Thus

$$\begin{aligned}
\frac{d}{d\gamma}|\xi(s,\chi)|^2 &= |\xi(s,\chi)|^2\frac{i}{2}\Big(\psi(\overline{1-s+\mathfrak{a}/2}) - \psi(1-s+\mathfrak{a}/2) + \psi(\overline{s+\mathfrak{a}/2}) - \psi(s+\mathfrak{a}/2)\Big) \\
&= |\xi(s,\chi)|^2\frac{i}{2}\Big(2i(\arg(\overline{1-s+\mathfrak{a}/2}) - \arg(s+\mathfrak{a}/2)) + O\big(\gamma^{-2}\big)\Big) \\
&\ll \gamma^{1-2\sigma}\Big(O\big(\gamma^{-1}\big) + O\big(\gamma^{-2}\big)\Big) \\
&\ll \gamma^{-2\sigma},
\end{aligned} \tag{34}$$

by Equation (6) and the Taylor expansion of arccot.

We use summation by parts, (34), (31), and (32) to estimate

$$\begin{aligned}
&\sum_{\gamma\le T}|\xi(\sigma+i\gamma,\chi_1)|^2\left|\sum_{n\le R}c_n n^{-\sigma-i\gamma}\right|^2\left|\sum_{n\le\sqrt{\frac{q\gamma}{2\pi l}}}\chi_1(n)n^{\sigma-1+i\gamma}\right|^2 \\
&\qquad = |\xi(\sigma+iT,\chi_1)|^2\sum_{\gamma\le T}\left|\sum_{n\le\sqrt{\frac{q\gamma}{2\pi l}}}\chi_1(n)n^{\sigma-1+i\gamma}\right|^2 - \\
&\qquad \int_1^T\sum_{\gamma\le t}\left|\sum_{n\le\sqrt{\frac{q\gamma}{2\pi l}}}\chi_1(n)n^{\sigma-1+i\gamma}\right|^2\frac{d}{dt}|\xi(\sigma+it,\chi_1)|^2\,\mathrm{d}t \\
&\qquad \ll T^{1-2\sigma}T^{\sigma-1/2}N(T) + \int_1^T t^{\sigma-1/2}N(t)t^{-2\sigma}\,\mathrm{d}t.
\end{aligned}$$

The first term is clearly $o(N(T))$, and in the integral we use the fact that $N(t) = O(t\log t)$ to estimate it as

$$\int_1^T t^{1/2-\sigma}\log t\,\mathrm{d}t \ll T^{3/2-\sigma+\epsilon}.$$

We thus get

$$\sum_{\gamma\le T}|\xi(\sigma+i\gamma,\chi_1)|^2\left|\sum_{n\le R}c_n n^{-\sigma-i\gamma}\right|^2\left|\sum_{n\le\sqrt{\frac{q\gamma}{2\pi l}}}\chi_1(n)n^{\sigma-1+i\gamma}\right|^2 \ll T^{3/2-\sigma+\epsilon} = o(N(T)),$$

$$\sum_{\gamma \le T} |\varphi(\sigma + i\gamma, \chi_2)|^2 \left| \sum_{n \le \frac{1}{R}\sqrt{\frac{l\gamma}{2\pi q}}} \chi_2(n) n^{\sigma - 1 - i\gamma} \right|^2 \ll T^{3/2-\sigma+\epsilon} = o(N(T)),$$

where the second estimate is obtained in the same way.

Finally we use the Cauchy-Schwarz inequality, (29), (30), and the above two equations to estimate all other terms in (27) as $o(N(T))$. □

5.3 Proposition 2

Since $B(s, P)$ is a finite Dirichlet polynomial, the series representation is valid everywhere and hence it is bounded independently of T. Thus, to estimate $\sum_{\gamma \le T} |A(\gamma)|^2$, it suffices to estimate

$$\sum_{\gamma \le T} |L(s, \chi_1)|^2 |L(s, \chi_2)|^2 \ll N(T). \tag{35}$$

The approximate functional equation for $L(s, \chi_1)$, as in the proof of Proposition 1, gives

$$\begin{aligned} L(s, \chi_1) &= \sum_{n \le \sqrt{\frac{qlt}{2\pi}}} \chi_1(n) n^{-s} + \xi(s, \chi_1) \sum_{n \le \sqrt{\frac{qt}{2\pi l}}} \overline{\chi_1(n)} n^{s-1} \\ &\quad + O\Big(t^{-\sigma/2} \log t\Big) + O\Big(t^{-1/4}\Big) \\ &= W_1 + \xi(s, \chi_1) W_2 + O\Big(t^{-\sigma/2} \log t\Big) + O\Big(t^{-1/4}\Big), \end{aligned}$$

$$L(s, \chi_2) = Y_1 + \xi(s, \chi_2) Y_2 + O\Big(t^{-\sigma/2} \log t\Big) + O\Big(t^{-1/4}\Big).$$

We have

$$\sum_{0 \le T} Y_1 \overline{Y_1} W_1 \overline{W_1} = \sum_{\gamma \le T} \sum_{m,n,\mu,\nu \le \sqrt{\frac{ql\gamma}{2\pi}}} \frac{\chi_1(m)\chi_2(n)\chi_1(\mu)\chi_2(\nu)}{(mn\mu\nu)^\sigma} \left(\frac{\mu\nu}{mn}\right)^{i\gamma}. \tag{36}$$

Again, we consider the diagonal terms separately from the rest of the sum. The number of solutions to $mn = \mu\nu = r$ is at most the square of the number of divisors of r, $d(r)^2$. Thus

$$\sum_{\gamma\leq T}\sum_{mn=\mu\nu}^{(q\gamma/2\pi)^{1/2}}\frac{\chi_1(m)\chi_2(n)\chi_1(\mu)\chi_2(\nu)}{(mn)^{2\sigma}}\ll\sum_{\gamma\leq T}\sum_{r=1}^{\infty}\frac{d(r)^2}{r^{2\sigma}}\ll N(T), \tag{37}$$

since the inner series converges. For the remaining sum set $\mu\nu=r$ and $mn=s$. We can treat the cases $s<r$ and $s>r$ separately. In the following analysis we assume m, n, μ, $\nu\leq(qlT/2\pi)^{1/2}$. Consider first the terms with $s<r$ in (36). We have that

$$Z_2=\sum_{r\leq qlT/2\pi}\sum_{s<r}\sum_{m|s,\mu|r}\frac{\chi_1(m)\chi_2(s/m)\chi_1(\mu)\chi_2(r/\mu)}{r^\sigma s^\sigma}\sum_{K\leq\gamma\leq T}\left(\frac{r}{s}\right)^{i\gamma}, \tag{38}$$

where $K=\min(T,(2\pi/ql)\max(m^2,s^2/m^2,\mu^2,r^2/\mu^2))$. Applying Landau's Formula, (3), to Z_2 gives

$$\begin{aligned}
Z_2&=\sum_{r\ll T}\sum_{s<r}\sum_{m|s,\mu|r}\frac{\chi_1(m)\chi_2(s/m)\chi_1(\mu)\chi_2(r/\mu)}{r^{\sigma+1/2}s^{\sigma-1/2}}\left(\sum_{\gamma\leq T}\left(\frac{r}{s}\right)^{\rho}-\sum_{\gamma<K}\left(\frac{r}{s}\right)^{\rho}\right)\\
&=\sum_{r\ll T}\sum_{s<r}\sum_{m|s,\mu|r}\frac{\chi_1(m)\chi_2(s/m)\chi_1(\mu)\chi_2(r/\mu)}{r^{\sigma+1/2}s^{\sigma-1/2}}\frac{K-T}{2\pi}\Lambda\left(\frac{r}{s}\right)\\
&+O\left(\sum_{r\ll T}\sum_{s<r}\sum_{m|s,\mu|r}\frac{1}{r^{\sigma+1/2}s^{\sigma-1/2}}\frac{r}{s}\log\frac{2Tr}{s}\ \log\log\frac{3r}{s}\right)\\
&+O\left(\sum_{r\leq cT}\sum_{s<r}\sum_{m|s,\mu|r}\frac{1}{r^{\sigma+1/2}s^{\sigma-1/2}}\log\frac{r}{s}\ \min\left(T,\frac{r/s}{\langle r/s\rangle}\right)\right)\\
&+O\left(\sum_{r\ll T}\sum_{s<r}\sum_{m|s,\mu|r}\frac{1}{r^{\sigma+1/2}s^{\sigma-1/2}}\log 2T\ \min\left(T,\frac{1}{\log {}^{r}/_{s}}\right)\right)\\
&=Z_{21,2}+Z_{23}+Z_{24}+Z_{25}.
\end{aligned}$$

For $Z_{21,2}$ we set $r=sk$. Recall the estimate $d(x)\ll x^\epsilon$. Now, since $K\leq T$ we get

$$\begin{aligned}
Z_{21,2}&\ll T\sum_{k\ll T}\sum_{s\ll T/k}\frac{\Lambda(k)k^\epsilon s^\epsilon}{k^{\sigma+1/2}s^{2\sigma}}\\
&\leq T\sum_{k=1}^{\infty}\frac{\Lambda(k)k^\epsilon}{k^{\sigma+1/2}}\sum_{s=1}^{\infty}\frac{1}{s^{2\sigma-\epsilon}}\\
&\ll T,
\end{aligned}$$

since both series converge. The Z_{23} is easy to estimate. We have

$$\begin{aligned}
Z_{23} &\ll \log T \ \log\log T \sum_{r\ll T} \frac{r^{\epsilon}}{r^{\sigma-1/2}} \sum_{s<r} \frac{s^{\epsilon}}{s^{\sigma+1/2}} \\
&\ll \log T \ \log\log T \sum_{r\ll T} \frac{r^{\epsilon}}{r^{\sigma-1/2}} \\
&\ll T^{3/2-\sigma+\epsilon} \log T \ \log\log T = o(N(T)),
\end{aligned}$$

because the series over s converges.

For Z_{24}, it is not surprising that we let $r = ls + t$, where $-s/2 < t \le s/2$. We will look at the two cases separately. Case 1 occurs when l is a prime power and $t \neq 0$, and case 2 happens otherwise. In case 2 we have by trivial estimates that

$$\begin{aligned}
&\ll \log T \sum_{s\ll T} \sum_{l\ll T/s} \sum_{|t|\le s/2} \frac{1}{(ls+t)^{\sigma-1/2} s^{\sigma+1/2}} \\
&\ll \log T \sum_{m,n,\mu,\nu} \frac{1}{(\mu\nu)^{\sigma-1/2}(mn)^{\sigma+1/2}} \\
&\ll T^{3/2-\sigma} \log T.
\end{aligned}$$

In case 1 we get that

$$\begin{aligned}
Z_{24} &\ll \sum_{s\ll T} \sum_{l<T} \sum_{0\neq|t|\le s/2} \frac{1}{(ls+t)^{\sigma-1/2} s^{\sigma-1/2} |t|} \log(l+t/s) \\
&\ll \log^2 T \sum_{s\ll T} s^{1-2\sigma} \sum_{l<T} l^{1/2-\sigma} \\
&\ll T^{7/2-3\sigma} \log^2 T.
\end{aligned}$$

From this we obtain the strict condition on σ. We need

$$\begin{aligned}
\frac{7}{2} - 3\sigma &< 1 \\
\frac{5}{2} &< 3\sigma \\
\frac{5}{6} &< \sigma,
\end{aligned}$$

so that we can estimate Z_{24} as $O(N(T))$.

It remains to estimate Z_{25}. We use the same method as in Proposition 1. Let

$s = r - k$, and $1 \leq k < r$ to get

$$\begin{aligned} Z_{25} &\ll \log T \sum_{r \ll T} \sum_{k<r} \frac{1}{r^{\sigma+1/2-\epsilon}(r-k)^{\sigma-1/2-\epsilon}} \frac{r}{k} \\ &\ll \log T \sum_{r \ll T} \frac{1}{r^{\sigma-1/2-\epsilon}} \sum_{k<r} \frac{1}{k} \\ &\ll T^{3/2-\sigma+\epsilon} \log^2 T = o(N(T)). \end{aligned}$$

Finally, if $s > r$ we can consider the complex conjugate of (36) to obtain the same estimate. The rest of the proof proceeds in the same way as in Proposition 1. We obtain trivially the estimates

$$\sum_{\gamma \leq T} |W_1|^4 \ll N(T), \tag{39}$$

$$\sum_{\gamma \leq T} |Y_1|^4 \ll N(T). \tag{40}$$

Also, by modifying the argument slightly we find that

$$\sum_{\gamma \leq T} |W_2|^4 \ll T^{2\sigma-1+\epsilon} N(T), \tag{41}$$

$$\sum_{\gamma \leq T} |Y_2|^4 \ll T^{2\sigma-1+\epsilon} N(T). \tag{42}$$

For example, if we let S be the first sum then

$$\begin{aligned} S &\ll \sum_{\gamma \leq T} \sum_{m,n,\mu,\nu} \frac{\chi_1(m)\chi_1(n)\chi_1(\mu)\chi_1(\nu)}{(mn\mu\nu)^{1-\sigma}} \Big(\frac{\mu\nu}{mn}\Big)^{i\gamma} \\ &= S_1 + S_2, \end{aligned}$$

where S_1 is the sum over diagonal terms. We find

$$\begin{aligned} S_1 &= \sum_{\gamma \leq T} \sum_{mn=\mu\nu}^{q\gamma/2\pi l} \frac{1}{(mn)^{2-2\sigma}} \\ &\ll \sum_{\gamma \leq T} \sum_{r=1}^{q\gamma/2\pi l} r^{2\sigma-2+\epsilon} \\ &\ll \sum_{\gamma \leq T} \gamma^{2\sigma-1+\epsilon} \\ &\ll T^{2\sigma-1+\epsilon} N(T). \end{aligned}$$

as required. We then proceed as before.

We also need to estimate the derivative of $|\xi(s,\chi)|^4$. By the estimate (34) from Proposition 1 we get

$$\begin{aligned}\frac{d}{d\gamma}|\xi(s,\chi)|^4 &\ll \gamma^{1-2\sigma}\gamma^{-2\sigma}\\ &\ll \gamma^{1-4\sigma}.\end{aligned}$$

The rest of the proof now follows from estimating

$$\sum_{\gamma\leq T}|\xi(s,\chi_1)|^4|W_2|^4$$

as

$$O\Big(T^{1-2\sigma+\epsilon}N(T)\Big),$$

and similarly for Y_2, and applying the Cauchy-Schwarz to the remaining terms in the expansion of the product in (35). □

5.4 Proposition 3

By Proposition 2 it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{d_n\chi_2(n)-e_n\chi_1(n)}{n^{2\sigma}}\neq 0. \tag{43}$$

First, we need to compute the d_n and e_n's explicitly. Let us denote the primes below P by $\mathscr{P}=\{p_1,p_2,\dots,p_h,P\}$. The coefficients d_n are defined by the Euler product

$$\prod_{p\leq P}(1-\chi_2(p)p^{-s})\times\prod_{p>P}\sum_{n=0}^{\infty}\frac{\chi_1(p^n)}{p^{ns}}.$$

We can see that no n with $p^2\mid n$, $p\in\mathscr{P}$ exists in the expansion. Hence for such n, $d_n=0$. If n has no prime factors from the set $\mathscr{P}$ then we just get the usual coefficient from $L(s,\chi_1)$. On the other hand, if some prime $p\in\mathscr{P}$ divides n exactly once then it contributes $-\chi_2(p)$. Hence

$$d_n=\begin{cases}\chi_1(n), & \text{if } p\nmid n \text{ for all } p\in\mathscr{P},\\ (-1)^k\chi_1\Big(\frac{n}{p_{i_1}\cdots p_{i_k}}\Big)\chi_2(p_{i_1}\dots p_{i_k}), & \text{if } p_{i_j}\parallel n \text{ for } p_{i_j}\in\mathscr{P} \text{ for all } j,\\ 0 & \text{otherwise.}\end{cases}$$

Similarly for e_n. Hence for $p > P$ the Euler factors of the two series in 43 are of the form

$$(1 - \chi_1(p)\chi_2(p)p^{-2\sigma})^{-1}$$

for both series. On the other hand, for $p \leq P$ we have

$$1 + d_p\chi_2(p)p^{-2\sigma} + d_{p^2}\chi_2(p^2)p^{-4\sigma} + \ldots = 1 - \chi_2^2(p)p^{-2\sigma},$$

and similarly for the second series. Now, if for all P we have

$$\prod_{p \leq P}(1 - \chi_2^2(p)p^{-2\sigma}) = \prod_{p \leq P}(1 - \chi_1^2(p)p^{-2\sigma}),$$

we get equality of individual terms by considering successive primes P and dividing the corresponding equations. This gives $\chi_1^2(p) = \chi_2^2(p)$ for all primes p. Thus, for example,

$$0 = \chi_1(q^2) = \chi_2(q^2),$$

which is a contradiction since q and l are distinct, that is, $\chi_2(q) \neq 0$. This completes the proof. $\square$

5.5 What next?

We discussed most of the assumptions of our Main Theorem in section 5.1. It remains to be noted that we restricted ourselves to the region $\sigma \in (\frac{5}{6}, 1)$. As the careful reader might have noticed, we only required $\sigma > 5/6$ in the Z_{24} term in Proposition 2. All of our other estimates work just fine even if $\sigma > 1/2$. There are a few directions to approach the problem with this term. One of them leads us to estimate

$$\sum_{m \leq X} \frac{\chi_2(m)U(m)}{m^s},$$

where $U(m)$ is defined by

$$U(m) = \sum_{\substack{n \leq X \\ nm < A}}^{\infty} \frac{\chi_1(n)\log n}{n^s},$$

where A is some positive constant. The problem with this approach is that it is unclear how one should take advantage of the fact that sums, such as

$$\sum_{n=1}^{\infty} \frac{\chi_1(n) \log n}{n^s},$$

have abscissa of convergence 0.

A slightly different approach would be to notice that the character sums we have in Z_{24} are actually Dirichlet convolutions, so the coefficients could be written simply as

$$(\chi_1 * \chi_2)(r)(\chi_1 * \chi_2)(s).$$

However, we cannot easily bound these as $O(1)$, so we are left with $O_\epsilon((rs)^\epsilon)$, which is not enough. The author is not aware of any results which would greatly improve this estimate. On the other hand one could attempt to keep these convolutions inside the sums over r and s, but even then the best estimate we can obtain is

$$\sum_{n \le X} (\chi_1 * \chi_2)(n) = O_\epsilon\Big(x^{1/3+\epsilon}\Big)$$

by [7, Theorem 4.16].

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