Exponential Sums Over Eigenvalues of Δ in $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$

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Introduction

- Classical case

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- Hyperbolic version

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- Some numerics

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- Proof via Selberg Trace Formula

ζ and Landau's Formula

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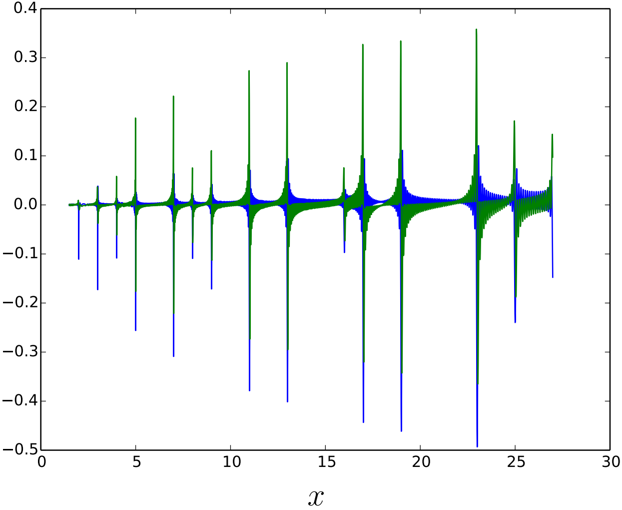
Landau's Formula (1912)

For a fixed $x > 1$,

$$\sum_{0<\gamma\leq T} x^{\rho} = -\frac{T}{2\pi}\Lambda(x) + O(\log T),$$

where $\Lambda(x)$ is the von Mangoldt function on $\mathbb{R}$.

Numerics on Landau's Formula



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- $\mathbb{H} = \{z = x + iy : x \in \mathbb{R}, y \in \mathbb{R}^+\}$, $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$

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Conjecture (Petridis–Risager 2014)

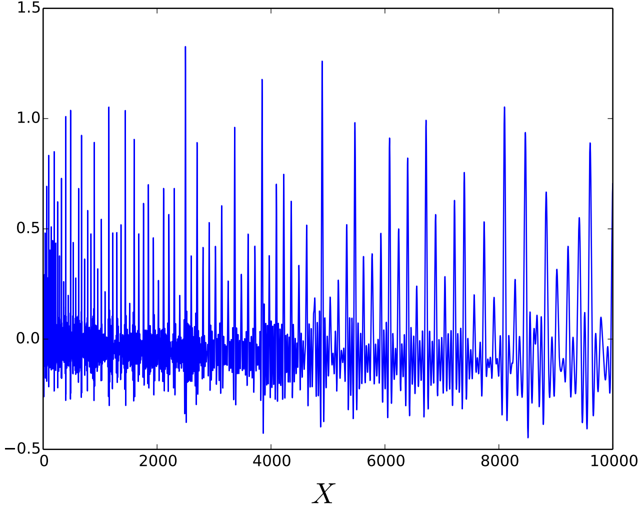
For $X > 1$,

$$S(T, X) \ll_\varepsilon T^{1+\varepsilon} X^\varepsilon.$$

X-plots

Numerical investigation with data by H. Then, A. Booker and A. Strömbergsson.

X-plots



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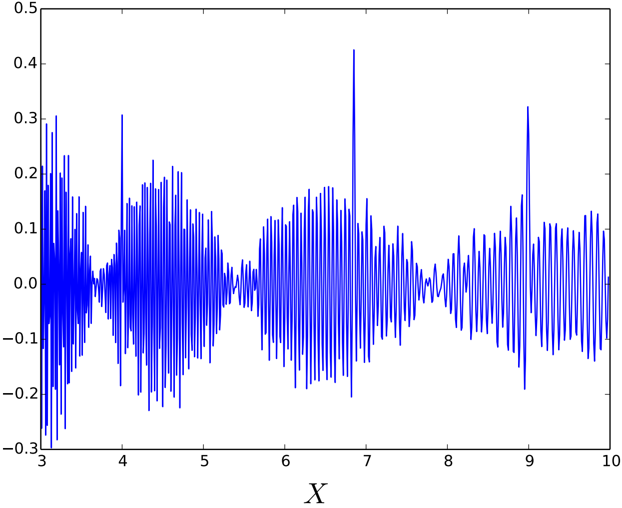

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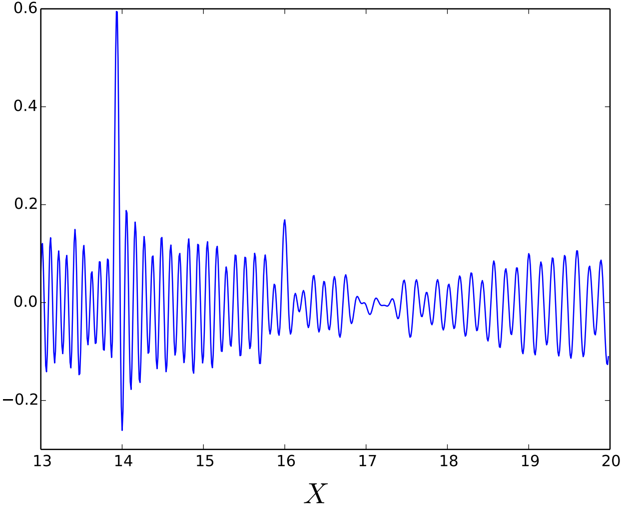
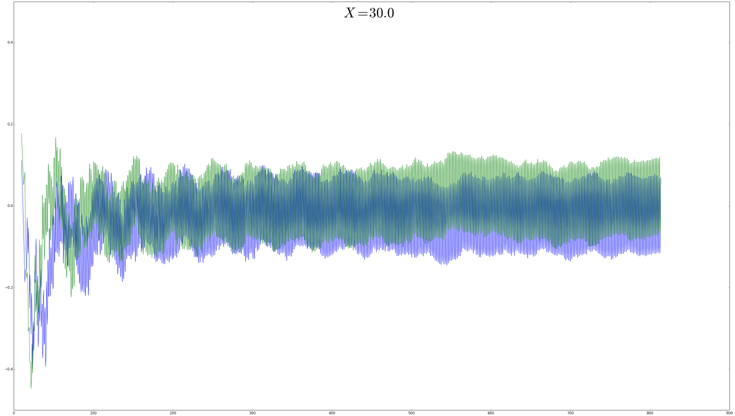


Animation!



The Result

Let $\Lambda_\Gamma(X) = \begin{cases} \log(N(\mathfrak{p})), & \text{if } X = N(\mathfrak{p})^\ell, \ell \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$

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For a fixed $X > 1$,

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as $T \to \infty$.

Proof

Selberg Trace Formula:

$$\sum_j h(t_j) + \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\frac{-\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)\,dr$$

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$+$ trash...

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Selberg Trace Formula: $h(r) = (\mathbb{1}_{[-T,T]} * \varphi_\epsilon)(r)\cos(r\log X)$

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Thanks!