Prime Geodesic Theorem in $\mathbb{H}^3$

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2D Setting

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- Suppose $\gamma \in \Gamma$ is hyperbolic: it has 2 fixed points on $\widehat{\mathbb{R}}$. The axis between these points gives a closed geodesic on M. This is invariant under conjugation.

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- Let $\lambda_j = s_j(1-s_j) = \frac{1}{4} + t_j^2$. Selberg:
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	$\ll X^{25/36+\epsilon}$	Soundararajan and Young (2013)

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- Too many eigenvalues (non-trivial zeros of Selberg zeta function).

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- $\mathrm{PSL}_2(\mathbb{C})$ acts on $\mathbb{H}^3$ with the action for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ given by

$$\gamma p = (ap + b)(cp + d)^{-1} \qquad \text{(taken in quaternions)}$$

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$$E_\Gamma(X) \ll X^{5/3+\epsilon}.$$

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- For $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$, Koyama, 2001 (conditionally):

$$E_\Gamma(X) \ll X^{11/7+\epsilon}.$$

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Theorem 2 (Chatzakos–Cherubini–L)

For general cofinite Kleinian group Γ, $V > 1$,

$$\frac{1}{V}\int_V^{2V} |E_\Gamma(x)|^2 dx \ll V^{16/5}(\log V)^{2/5}.$$

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Lemma

Let u_j be a Maass-Hecke cusp form on M, then

$$\sum_{r_j \leq T} \frac{r_j}{\sinh \pi r_j} \left| L(\tfrac{1}{2} + it, u_j \otimes u_j) \right| \ll T^{7/2+\epsilon} |t|^{1+\epsilon}.$$

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In the proof we make use of a spectral large sieve by Watt (2014).

In the theme of large sieves...

Theorem (Spectral Large Sieve (Watt 2014))

Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$, $T, N \gg q$ *and* $\{a_n\}$ *a sequence with* $a_n \in \mathbb{C}$. *Then*

$$\sum_{r_j \leq T} \frac{r_j}{\sinh \pi r_j} \left| \sum_{N < |n|^2 \leq 2N} a_n \rho_j(n) \right|^2 \ll \left(T^3 + T^{3/2} N^{1+\epsilon} \right) \|a_N\|^2.$$

Summary of proof of Theorem 1 (cont.)

We can show that

$$\sum_{n \in \mathcal{O}} \frac{r_j f(|n|) |\rho_j(n)|^2}{\sinh \pi r_j} = cN + O\left(\frac{N^{1/2} r_j}{\sinh \pi r_j} \int_0^\infty \frac{|L(\frac{1}{2} + it, u_j \otimes u_j)|}{(1 + |t|)^p} dt \right).$$

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Sum over j to get

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Some immediate corollaries

Corollary

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Corollary

For every $\epsilon > 0$, we have

$$\sum_{\substack{d \in \mathscr{D} \\ |\epsilon_d| \leq X}} h(d) = \mathrm{Li}(X^4) + O(X^{13/4+\epsilon}).$$

Here $\mathscr{D}$ is the set of discriminants of binary quadratic forms over $\mathbb{Z}[i]$.

Thank you!