Applications of Landau's Formula

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Introduction

- Landau and his work

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- Application to Dirichlet L-functions

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- Hyperbolic Landau-type formula

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Theorem (Landau's Formula)

For a fixed $x > 1$,

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The von Mangoldt function $\Lambda(x) = \begin{cases} \log p, & \text{if } x = p^n, \\ 0, & \text{otherwise.} \end{cases}$

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Theorem (Gonek–Landau Formula (1993))

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Let x, $T > 1$. Then

$$\sum_{0<\gamma\leq T} x^{\rho} = -\frac{T}{2\pi}\Lambda(x) + O(x \log 2xT \log\log 3x)$$

$$+O\left(\log x \min\left(T, \frac{x}{\langle x\rangle}\right)\right) + O\left(\log 2T \min\left(T, \frac{1}{\log x}\right)\right),$$

where $\langle x\rangle$ denotes the distance from x to the nearest prime power other than x itself.

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where $\langle x\rangle$ *denotes the distance from* x *to the nearest prime power other than* x *itself.*

- This is uniform in x!

Results of Gonek

Corollary 1 (Gonek)

Under the RH, for large T and α real with $|\alpha| \leq \frac{1}{2\pi}\log T$, we have

$$\sum_{0<\gamma\leq T} |\zeta(\tfrac{1}{2} + i(\gamma + 2\pi\alpha/\log T)|^2 =$$
$$\left(1 - \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2\right) \frac{T}{2\pi}\log^2 T + O\left(T \log^{7/4} T\right).$$

Results of Gonek II

Corollary 2 (Gonek)

Assume the RH and fix $0 < \alpha < 1$. Then as $T \to \infty$

$$\sum_{0<\gamma,\gamma'\leq T} \left(\frac{\sin(\frac{\alpha}{2}(\gamma - \gamma') \log T)}{\frac{\alpha}{2}(\gamma - \gamma') \log T} \right)^2 \sim \left(\frac{1}{\alpha} + \frac{\alpha}{3} \right) \frac{T}{2\pi} \log T.$$

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- R. Murty and K. Murty (1994): If $F, G \in \mathcal{S}$ then

$$|Z_F(T)\Delta Z_G(T)| = o(T) \implies F = G,$$

where $Z_F(T)$ is the set of zeros of $F(s)$ in $\mathrm{Re}(s) \geq 1/2$, $|\,\mathrm{Im}(s) \leq T$, Δ is the symmetric difference.

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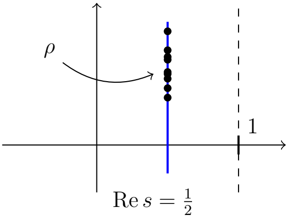

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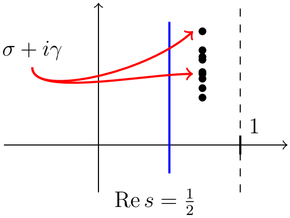

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Theorem 2 (L–Petridis 2012)

Two Dirichlet L*-functions with distinct primitive non-principal characters attain different values at* cT *non-trivial zeros of* ζ *up to height* T, *for some positive constant* c.

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- Not dependent on RH

Proof of Theorem 1

Notice z, $w \in \mathbb{C}$ are L.I. over $\mathbb{R}$ iff $|z\overline{w} - \overline{z}w| > 0$.

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then

$$\sum_{\substack{0<\gamma\leq T \\ A(\gamma)\neq 0}} 1 \overset{\text{C-S}}{\geq} \frac{|\sum A(\gamma)|^2}{\sum |A(\gamma)|^2} \gg \frac{|c|^2 N(T)^2}{N(T)} = |c|^2 N(T).$$

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- The 4th moment is harder to estimate but we only need an upper bound

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Similar to Theorem 1, but we start from

$$\frac{1}{2\pi i}\int_{\mathfrak{C}}\frac{\zeta'}{\zeta}(s)B(s,p)L(s,\chi)\,ds.$$

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- The $B(s,p)$ factor is now simpler which makes proving $c \neq 0$ easier (so we get complex characters)
- A more *natural* approach, but the analysis is more difficult (Garunkstis–Kalpokas–Steuding, 2010)

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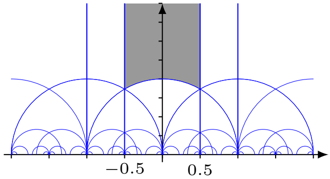

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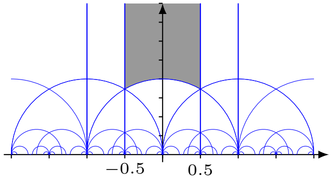

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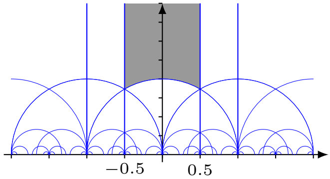

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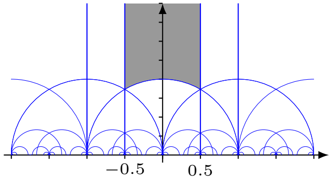

- $f : \mathbb{H} \longrightarrow \mathbb{C}$ is called an *automorphic form* if
  - $f(\gamma z) = f(z)$, for all $\gamma \in \Gamma$
  - $(\Delta + \lambda) f = 0$, where Δ is the Laplacian on $\mathbb{H}$

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- Eisenstein series:

$$E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\mathrm{Im}(\gamma z))^s$$

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Theorem (Prime Geodesic Theorem)

Let $\pi_\Gamma(x) = \#\big\{\{\gamma\} : N(\gamma) \leq x\big\}$, *then* $\pi_\Gamma(x) \sim \frac{x}{\log x}$.

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- Apply pre-trace formula to a mollified version of

$$\chi_{[0,(X-2)/4]}(u)$$

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$$N_f(X) = \int_{\Gamma\backslash\mathbb{H}} f(z)N(z,z,X)\,d\mu(z)$$

- Apply pre-trace formula to a mollified version of

$$\chi_{[0,(X-2)/4]}(u)$$

- $S(T,X)$ appears when estimating the contribution of the discrete spectrum

The Conjecture

Conjecture (Petridis–Risager 2014)

For $X > 1$,

$$S(T, X) \ll_\varepsilon T^{1+\varepsilon} X^\varepsilon.$$

- This implies the best possible error term in PGT $O(x^{3/4+\epsilon})$

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- Also for hyperbolic lattice counting (on average) $O(x^{1/2+\epsilon})$

Landau-type Formula

Let $\Lambda_\Gamma(X) = \begin{cases} \log(N(\mathfrak{p})), & \text{if } X = N(\mathfrak{p})^\ell, \ell \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$

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Theorem 3 (L 2014)

For a fixed $X > 1$,

$$S(T, X) = \frac{|F|}{\pi} \frac{\sin(T \log X)}{\log X} T + \frac{T}{\pi}(X^{1/2} - X^{-1/2})^{-1} \Lambda_\Gamma(X) + \frac{2T}{\pi} X^{-1/2} \Lambda(X^{1/2}) + O\left(\frac{T}{\log T}\right)$$

Proof

By Selberg Trace Formula:

$$\sum_{t_j>0} h(t_j) + \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\frac{-\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)\,dr$$

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$+$ trash...

Proof (cont.)

- Set $h(r) = (\chi_{[-T,T]} * \psi_\epsilon)(r)(X^{ir} + X^{-ir})$

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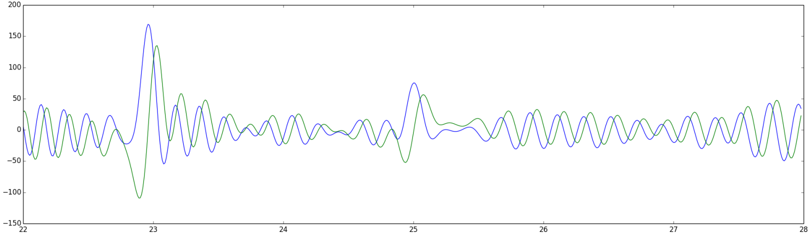
- Trickiest term to estimate is the contribution of continuous spectrum
- However, after a suitable change of contour we can use an argument similar to Landau's original one

Remarks (Theorem 3)



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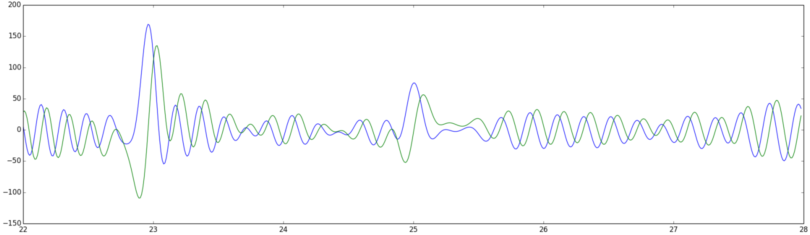

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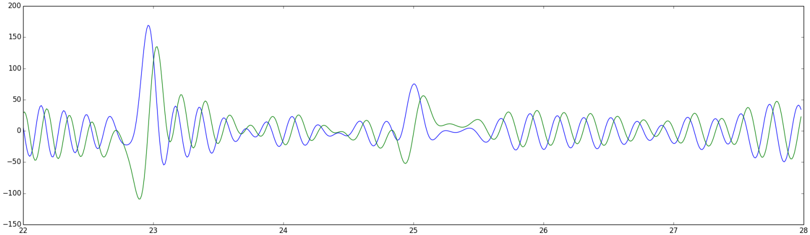

- We would like an actual Landau formula!
- Generalisation to other spaces?

Kiitos!
ありがとうございます