Quantum Chaos and the Riemann Hypothesis

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It is at the same time unnerving and awe-inspiring that the world of tiny particles should have anything to do with the mysteries of the prime numbers. Does nature really contain the clues to unravel the dilemma? It requires us to look at a story, which spans over three centuries, to understand how we have ended up in such a situation. Unfortunately, this story doesn't yet have an end, and in fact has the most powerful cliffhanger of all times. The accidental encounter of two people over tea, however, might just have changed this. Come to find out why.

1 What Is Chaos?

Classically — for systems obeying Newton's 2nd Law — chaos is often understood as extreme sensitivity to small changes in initial conditions. This is sometimes colloquially known as the *butterfly effect*: even the very small flapping of wings of a butterfly can eventually lead the system of global weather to produce a tornado in some other location. This is not a good example though. After all, the weather is a very complicated system so of course we expect bizarre behaviour. What is more striking is that it is often systems governed by very simple rules which exhibit chaos. We'll see two examples of this in a second. But first it is important to make a distinction between chaos and randomness.

You may be familiar with the concept of random processes from your probability course. This is not what chaos is at all. Random is truly *random*. Chaotic systems, however, are governed by deterministic rules[1] so there is no chance for anything to be random. What we do experience instead is the impossibility of predicting the behaviour of the system. Yet, through chaos we often end up observing a certain level of order, which hints that there's more going on behind the scenes than what's been revealed. This is not a mere figure of speech, however, as you can witness from the following example which depicts the so called *Lorenz attractor*. It is a chaotic system that arises when modelling weather on a large scale, for example, and still it is easy to see that there is a certain amount of order present (as we might expect!).

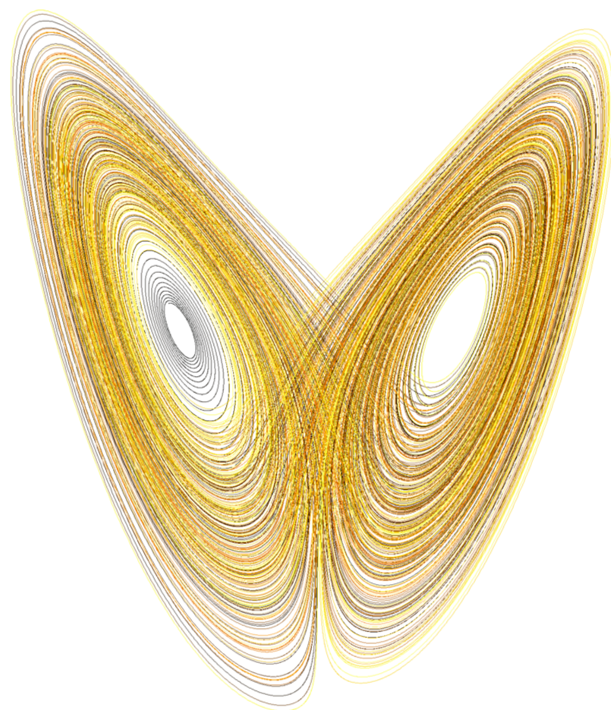

Figure 1: Lorenz Attractor. Source: Wikimedia Commons

Why does this happen though? Where does this chaos mathematically arise from? It turns out that chaos requires the equations describing the dynamical system to be nonlinear. For example, the Lorenz attractor is given by the system

$$\frac{dx}{dt} = \sigma(y - x), \qquad \frac{dy}{dt} = x(\rho - z) - y, \qquad \frac{dz}{dt} = xy - \beta z.$$

[1]At least in our case. There are things known as Complex Systems where you don't require deterministic rules. These systems end up exhibiting similar behaviour to chaos and are not random at all, despite what one might expect.

Let's look at another example, the logistic map. It is a discrete recurrence relation given by

$$x_{n+1} = rx_n(1 - x_n),$$

where $r > 0$. These equations can describe for example the evolution of a population with r being the rate of reproduction. Up to about $r = 3.57$ or so this relation yields a nice graph with predictable behaviour. Around this boundary all hell breaks loose. Even the slightest change to r dramatically changes the look of the graph. Yet, this is not the whole story: we can still find ranges of $r > 3.57$ where we *do* get regular behaviour. These ranges are called *islands of stability*.

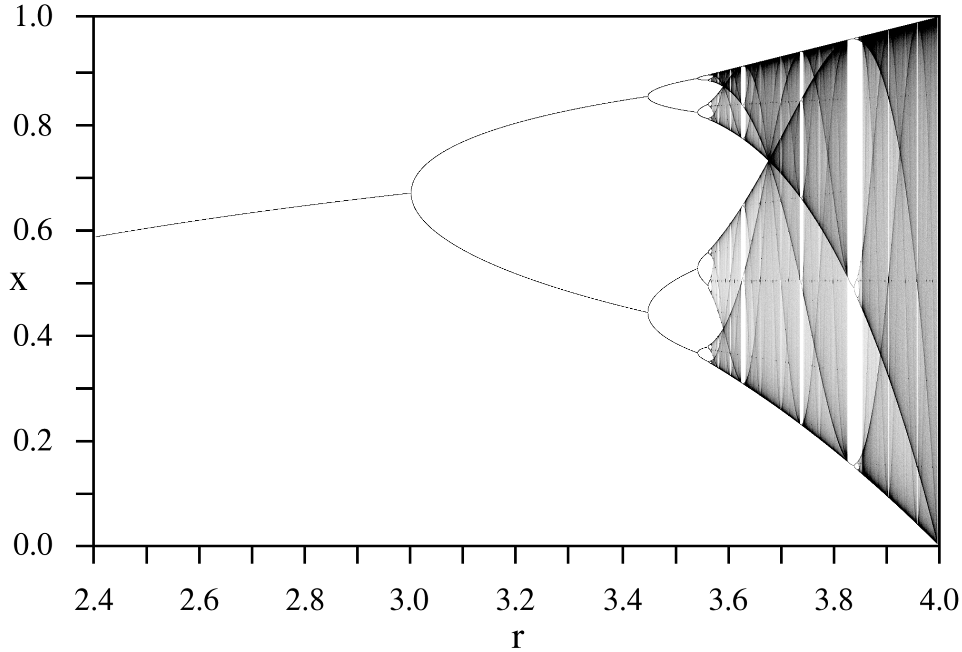

Figure 2: Bifurcation diagram for the logistic map. It displays the amount of periodic orbits present in the system for different r. Source: Wikimedia Commons

One final example which we will return to later in a different setting: dynamical billiards. You should already be familiar with these from last term's talk, but just in case let me start from the beginning. As a dynamical system we can think of a dynamical billiard as a region Ω in $\mathbb{R}^2$ such that the potential V of the system satisfies $V(\Omega) = 0$, $V(\mathbb{R}^2 \setminus \Omega) = \infty$. Moreover we place a single particle inside the region with a specific initial velocity and direction. After that the evolution of the system is governed by the law that angle of reflection equals angle of incidence and no energy is lost. The shape of the billiard table obviously affects the motion. A rectangular table gives very systematic motion whereas a stadium shaped one already yields chaos.

If you want to learn more about chaos or complex systems I suggest the following texts: Thompson and Stewart [7], Batty [2], Barabási [1].

2 Quantum Mechanics

In case you haven't encountered quantum mechanics before, I will give a quick primer containing the bare minimum needed for this talk. For a very approachable introduction see Griffiths and Harris [4].

In classical mechanics you are used to being able to describe all quantities of a system accurately. More precisely, what I mean is that if you have a swinging pendulum, for example, you can at any given time determine its exact position, velocity and so on. In quantum mechanics, however, this is fundamentally[2] impossible. The more accurately you try to measure a particle's position the harder it becomes to measure its velocity (or momentum). This is called the Heisenberg Uncertainty Principle and is sometimes expressed in the form $\sigma_x \sigma_p \geq \frac{\hbar}{2}$, where the σ's denote the usual statistical quantity of standard deviation. More accurately then, this says that the more accurate our measurement of, say, the position of the particle is (i.e. the standard deviation is small) the larger the standard deviation for the momentum must be.

This is exactly analogous to the situation of a wave on a string. If we flip the string in such a way that only a single moving wave is produced then it is pretty easy to tell the position of that wave, but we don't really have any meaningful wavelength for a lone wave. Whereas if we start flipping the string in a more controlled manner and produce sinusoidal waves then the wavelength is straightforward to determine, but it becomes harder to state where the wave actually *is* at any given time. This analogue is valid because of the de Broglie formula which relates the momentum of a particle to the wavelength of its wave function.

All of quantum mechanics revolves around (get it?) the *Schrödinger equation* for the

[2] That is, *provable* from the postulates of quantum mechanics.

wave function $\Psi(x,t)$, which is usually given as

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + \Psi V, \tag{1}$$

where V is the potential energy of the system, ∇^2 is the Laplacian, and $\hbar$ denotes the reduced Planck's constant. Equation (1) governs the evolution of the quantum system and the wave function describes the state of the system at any given time as a *function of the position of the particle(s)* in such a way that $|\Psi(x,t)|^2\, dx$ gives the probability of finding the particle at point x at time t. What physicists like to do is look for certain kinds of solutions. With wave-like problems it often happens that combinations of stationary solutions (not dependent on time) yield all the possible solutions to the system. Moreover, the stationary solutions conveniently arise from separable wave functions. Thus, we may write $\Psi(x,t) = \psi(x)f(t)$, where ψ is called the time-independent wave function. If we put these into (1) we find that the time evolution of a quantum systems is always of the form $f(t) = e^{-iE_nt/\hbar}$, and that ψ satisfies the *time-independent Schrödinger equation*

$$\hat{H}\psi_n = E_n\psi_n, \tag{2}$$

where $\hat{H}$ denotes the Hamiltonian operator of the system, $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V$. This is clearly an eigenvalue problem for $\hat{H}$! The importance of this comes from the fact that $\hat{H}$ is Hermitian and so its eigenvalues are *real*. Moreover, the eigenvalues E_n are the allowed energies of the particle.[3] In particular, for each stationary solution there is only one allowed energy level! This is in stark contrast to classical systems which allow continuous energy levels.

Now back to chaos. Have a look at what we said about chaos, now stare at the Schrödinger equation, now back to me. Notice anything? Indeed, Schrödinger's equation is *linear*, which means that there can be no such thing as quantum chaos! Great, can we all go home now? Unfortunately, this is just a result of indecisiveness among the physicists. When they were originally looking for chaos in the quantum world they realised that it is impossible to have chaos in the classical sense, exactly because of the above reason. This opened up a debate: what should we call this thing we are researching if it doesn't even exist? Sometimes people get boggled by trivialities. Nevertheless, various alternatives were suggested, such as Quantum Chaology. Eventually everyone came up with their

[3] In this talk I use "system" and "particle" synonymously since I will only consider quantum systems with one particle.

own favourite and no one preferred anyone else's, so they settled back on quantum chaos. Ain't that just nice?

What do we understand by quantum chaos then? You might know that quantum mechanics was developed to in a way "replace" the classical Newtonian mechanics. Therefore, quantum mechanics should be able to account for the behaviour of the world on the larger "Newtonian" scale as well. This is the Bohr Correspondence Principle and it happens when $\hbar \longrightarrow 0$, the so-called semiclassical limit. Quantum chaos then is the study of systems whose semiclassical limit exhibits chaos. So, for example, the study of quantum billiards whose classical versions are chaotic falls within the realm of quantum chaos.

3 Interlude to Analytic Number Theory

We'll now cover some more mathematical ground. I will outline enough material about analytic number theory to gain a basic understanding of the Riemann Hypothesis. We'll start with the most central definition in this field.

Definition 1. *The Riemann zeta function is defined for real $s > 1$ by the absolutely convergent series*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Let us go back in time to when quantum mechanics was still as much out of reach as getting the Brits to understand the word "insulation" is today. It's no surprise that this takes us all the way back to *the man* himself — Professor Leonhard Euler from the lovely land of the Alps — in the year 1737. At the time he was a professor and the chair of the mathematics department at the Academy of Sciences in St. Petersburg, Russia, having wed his wife only 3 years previously. During this year, in which the famous University of Göttingen[4] was founded, Euler published a result which revealed the first connection between the prime numbers and the zeta function — a finding which would eventually spark centuries of intense research. Euler proved that the zeta function has an alternate

[4] This is where modern analytic number theory got its foundations.

expression as a product over all primes:

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1}$$

Notice, that this formula contains a proof of the infinitude of primes. As $s \longrightarrow 1$, ζ approaches the divergent harmonic series. Now, if there were only finitely many primes the product would remain finite as we pass to the limit, but this is a contradiction. This argument is very different from the classical one given by Euclid and it already hints at the deep underlying coupling between the zeta function and the primes. It is as if somehow this uncannily simple function somehow manages to encode almost any possible information about the mysterious prime numbers. We will see more evidence of this later.

Let us fast forward a hundred years or so to 1859, during which we witness the rise and fall of such mathematical giants as Carl Friedrich Gauss and Évariste Galois. We zoom to the aforementioned city of Göttingen in Germany into the office of a clever chap called Bernhard Riemann, who had also just become head of the mathematics department there. He must have been quite happy with himself having just published a paper in number theory despite being a mathematical physicist. Little did he know at the time, though, that the innocent assumption slipped in between some of the proofs would wreak havoc amongst the greatest minds the world would have to offer in the future.

In order to understand what he said, we need to look at some of the other ideas in his paper. Inspired by the findings of Euler described before, Riemann was led to investigate the aptly named Riemann zeta function. Cauchy & Co. had just finished laying the foundations of complex analysis and Riemann took advantage of this. His key idea was to notice that we can consider ζ as a function of a complex variable, and that it does in fact converge for all $\operatorname{Re} s > 1$. Moreover, in this region ζ defines an analytic function, and we all know that analytic functions are *nice*. Among the few proofs in his paper (most of it consisted of definitions, conjectures, sketches of proofs, etc.) is another stroke of genius known as the *functional equation* for ζ, which says that

$$\zeta(s) = \pi^{s-1/2} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \zeta(1-s).$$

This equation yields an extension of ζ to all of $\mathbb{C}$ as a meromorphic function with a simple pole of residue 1 at 1.

If you came to my previous talk on Nevanlinna Theory[5] you know that the zeros of analytic or meromorphic functions are very important. So let's have a look at the zeros of ζ. From the Euler Product, which is clearly valid wherever the series converges, we can deduce that $\zeta(s) \neq 0$ for $\operatorname{Re} s > 1$ (otherwise one of the terms in the product should be 0, which is not the case). On the other hand, we know that the Gamma function has simple poles at the negative integers and at 0. The functional equation then implies that ζ has simple zeros at the negative even integers (not at 0 though, $\zeta(0) = -1/2$), and no more zeros in this region. These are the so called "trivial zeros" of the zeta function, and they are well understood. This leaves us with the "critical strip" $0 < \operatorname{Re} s < 1$. Riemann argued that in this region ζ has infinitely many zeros. Let us denote these zeros by Z. Moreover, from the functional equation and the simple fact that $\overline{\zeta(s)} = \zeta(\overline{s})$, he concluded that the zeros satisfy a strong symmetry. Namely, if $\zeta(\rho) = 0$ then $\overline{\rho}$, $1 - \rho$, and $1 - \overline{\rho}$ are zeros as well. The important thing to notice here is that it is the functional equation which implies the symmetry of the zeros about the line $\operatorname{Re} s = 1/2$. This line is called the "critical line" of ζ. We are now in a position to state the hypothesis.

Riemann Hypothesis.

$$Z \subset \{s \in \mathbb{C} : \operatorname{Re} s = 1/2\}.$$

So Riemann said that all the zeros in fact lie on the axis of symmetry. He couldn't prove it though, and neither could anyone else. At first glance the way we stated the hypothesis might seem a little innocent. Do not be deceived though, if true this result would have wide ranging consequences.

At the time of Riemann one of the most important consequences of the Riemann Hypothesis would have been the sharpest possible estimate in the Prime Number Theorem, which was still unsolved. Recall that the Prime Number Theorem says that $\pi(x) \sim \operatorname{li}(x)$ where $\pi(x)$ is the number of primes below x and $\operatorname{li}(x)$ is the logarithmic integral $\operatorname{li}(x) = \int_2^x \log^{-1} t \, \mathrm{d}t$. The Riemann Hypothesis would give us the error term to be $O\left(x^{1/2} \log^2 x\right)$. This is where the initial interest came from. Since then there have been a lot of stories about attempted solutions[6]. Yet, the fortress Riemann still holds.

[5] http://ucl.sneffel.com/index.php/File:Laaksonen-nev-26Oct.pdf

[6] Have a look at http://www.ams.org/notices/200303/fea-conrey-web.pdf

4 Fitting It All Together — The Story of Two Men

We keep going along these lines for another hundred years or so: many grand ideas, little to no success. In the beginning of the 20th century we finally see the seeds of our main idea being planted. It was noticed that since the Riemann zeros are predicted to lie on a line and the energy levels of a quantum mechanical system are always real (and hence lie on a line), there could be some kind of relationship between them. George Pólya and David Hilbert both suggested that Riemann zeros would arise as the eigenvalues of some Hermitian operator — the so called Hilbert-Pólya conjecture. This is the so called spectral theoretic approach. At the time, however, there was little evidence of this being the case and not much happened on this front for a while.

Moving on, we arrive to the year 1972 and another continent. We are looking at two guys, a physicist Freeman Dyson and a number theorist Hugh Montgomery, sipping their afternoon tea at the Institute for Advanced Study at Princeton, USA. Montgomery was explaining his recent work to his companion when Dyson suddenly remarks "Extraordinary! Do you realize that's the pair-correlation function for the eigenvalues of a random Hermitian matrix? It's also a model of the energy levels in a heavy nucleus"[7].

Both of them had been working on something known as pair-correlation — Montgomery for the Riemann zeros and Dyson for the eigenvalues of random matrices. Montgomery had conjectured that the zeros of ζ cannot actually occur too close to one another. More precisely, if we assume the Riemann hypothesis then the proportion of zeros separated by a magnitude u is given by

$$1 - \left(\frac{\sin \pi u}{\pi u}\right)^2 .$$

Strikingly this same formula describes the spacings of energy levels in specific quantum mechanical systems — quantum *chaotic* systems!

You might have noticed the term "random matrix" in the quote by Dyson. For the sake of keeping these notes concise I will only briefly touch upon this. Random matrices, of course, are matrices with random entries with a specific probability distribution. It suffices to say that it is one of the main tools used to study the energy levels of quantum

[7] `http://www.americanscientist.org/issues/pub/the-spectrum-of-riemannium`

mechanical systems. This tool has subsequently entered the world of pure mathematics as well.

As the final words I wish to outline the subsequent development in the quest to conquer the fortress of Riemann. So, as said before, the pair-correlation statistic of the zeta function seems to correspond to the aforementioned energy levels. This is actually because these energy levels are described by eigenvalues of random Hermitian matrices. So we can limit ourself to studying this connection between random matrix theory (RMT) and the Riemann zeta function. It has been conjectured, the so called GUE conjecture, that other statistics, such as the nearest-neighbour distribution[8], of the Riemann zeros could correspond to those of random matrices. It has actually been shown that this should be the case through substantial calculations carried out by Odlyzko. This serves to give more evidence for the connection between ζ and quantum chaos.

For now, I want to leave all the ideas mentioned here hanging in the air, slowly sinking into the reader's mind hopefully revealing the profoundness of this discussion. It is suggested that those intrigued to read further consult the following texts: [6], [3], [5], and the references therein.

[8]This is in contrast to the pair-correlation which considers all zeros at the same time. In nearest-neighbour one considers the spacings between successive zeros.

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