UCL

Prime Number Theorem

Niko Laaksonen
UCL

Image courtesy of Wikimedia

Of Life and Death

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- Cicada

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- Life cycles in predator–prey system

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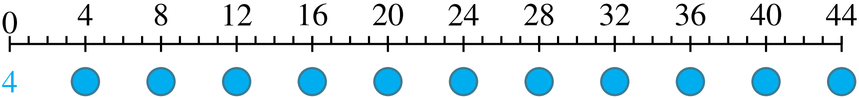

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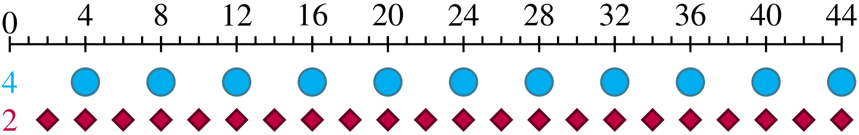

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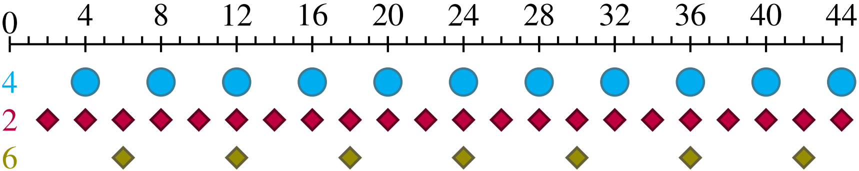

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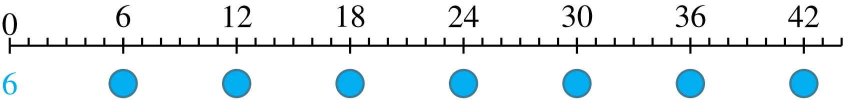

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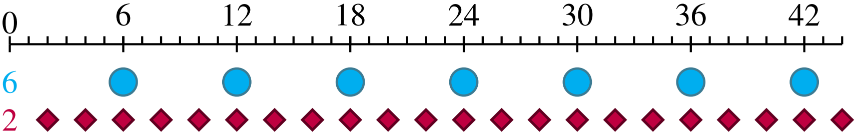

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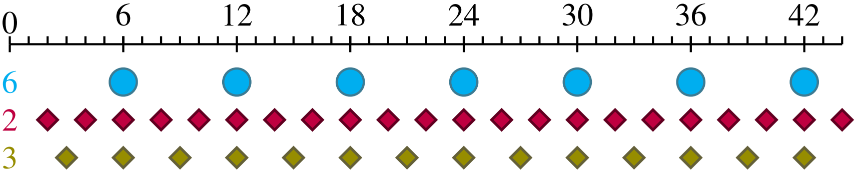

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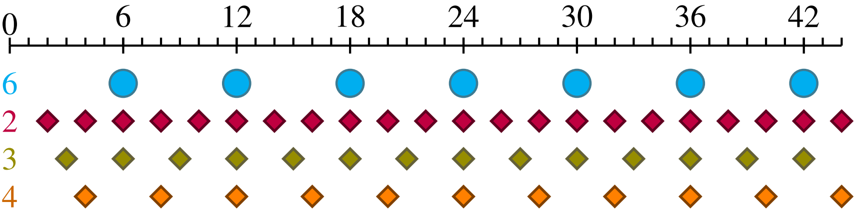

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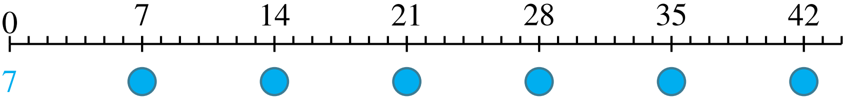

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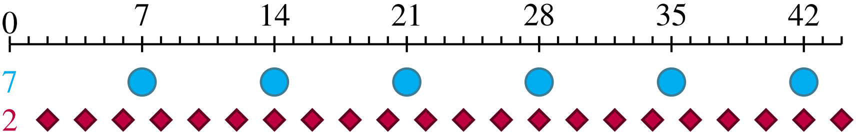

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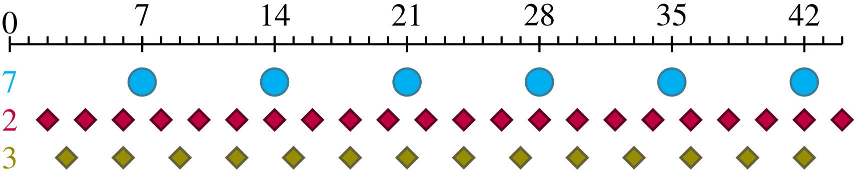

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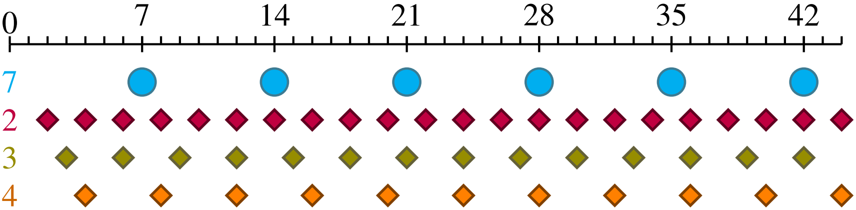

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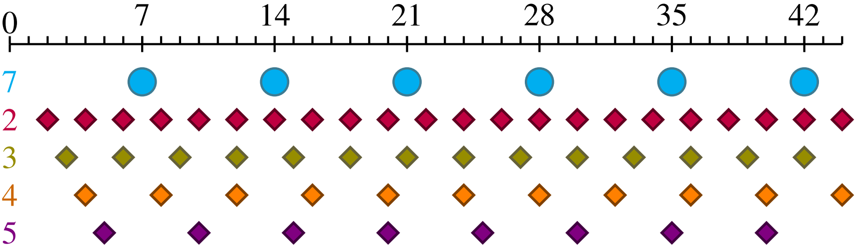

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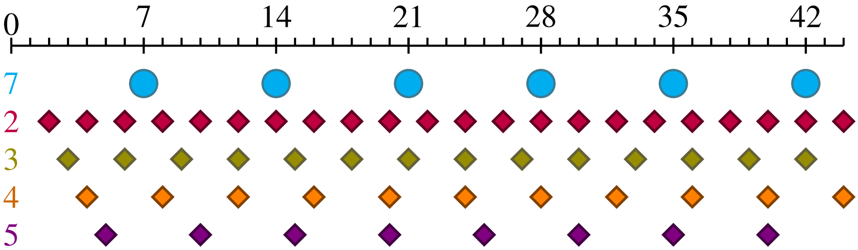

7 has no non-trivial divisors!

Prime Number Facts

Definition

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- However, $1\,033\,333 = 7 \times 43 \times 3433$.
- In fact every number can be written as a product of primes.
    - This representation is unique.

How many primes are there?

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There are infinitely many primes.

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- Largest publicly known prime is $2^{43112609} - 1$

So how do we find them?

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Sieve of Eratosthenes (ca. 200 BC)

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Consider the numbers $N!+2$, $N!+3$, $\ldots$, $N!+N$.

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In other words, there are abritrarily long gaps between primes.

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- How many primes up to some number x?

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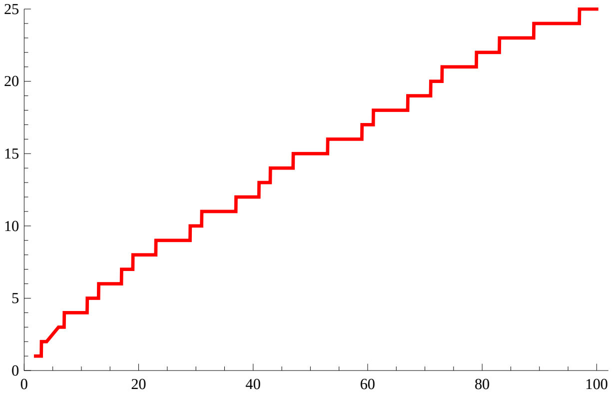
- Can we find a formula for $\pi(x)$?
- Not a smooth function—step function
- Can we find $f(x)$ such that $f(n) = \pi(n)$ if n is a prime?
- i.e. Can we approximate $\pi(x)$ by a smooth function $f(x)$?

A Formula for Primes



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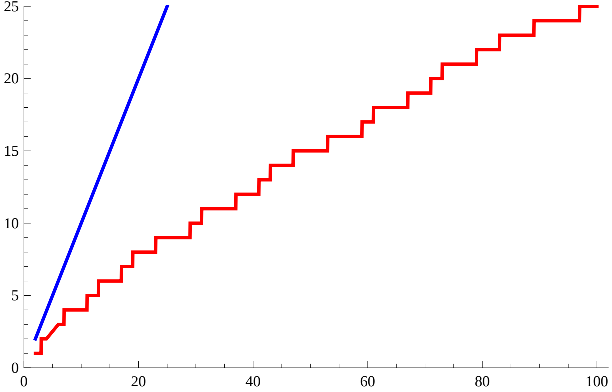

$$f(x) = x$$

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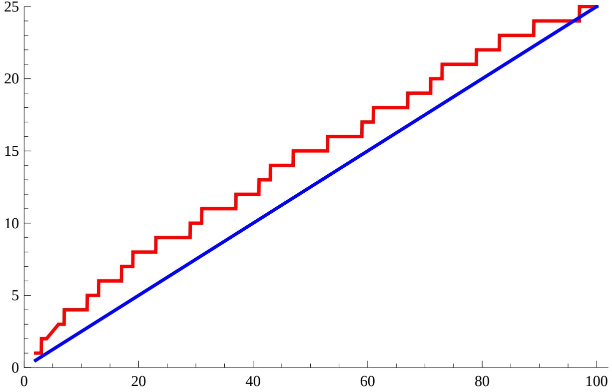
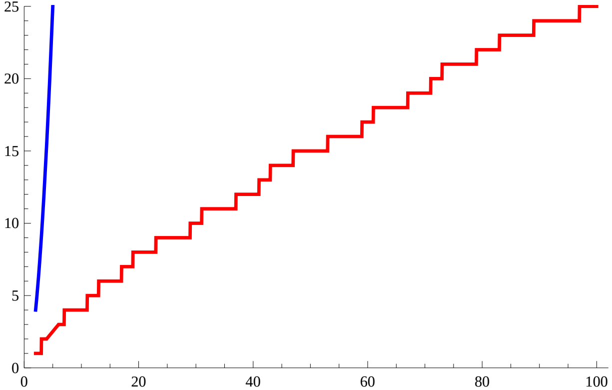


$$f(x) = \frac{x}{4}$$

A Formula for Primes



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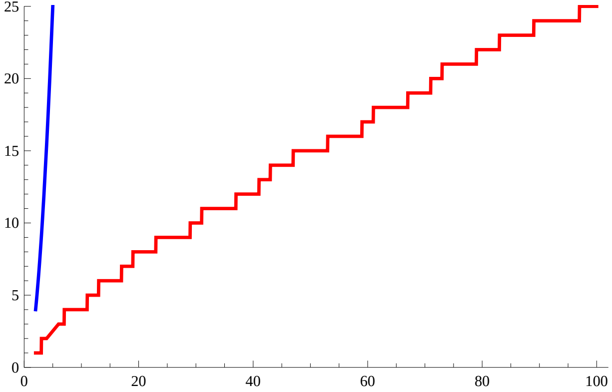

$$f(x) = x^2$$

In fact, no polynomial will do!

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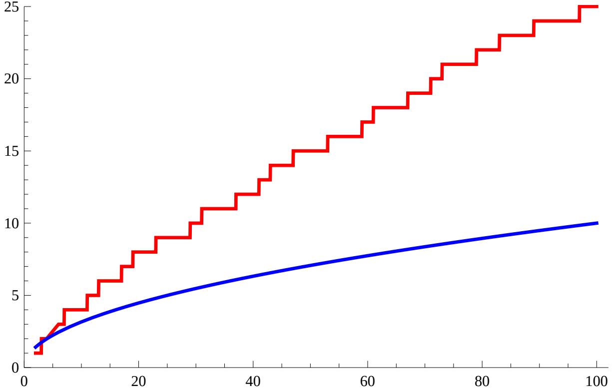

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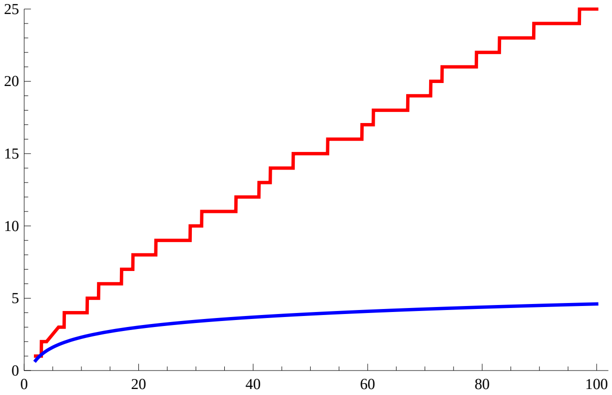

$$f(x) = \log x$$

Gauß (1777–1855)

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- Born to a poor family

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- Child prodigy

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The ratio gets closer to 1 as x gets bigger. Great!

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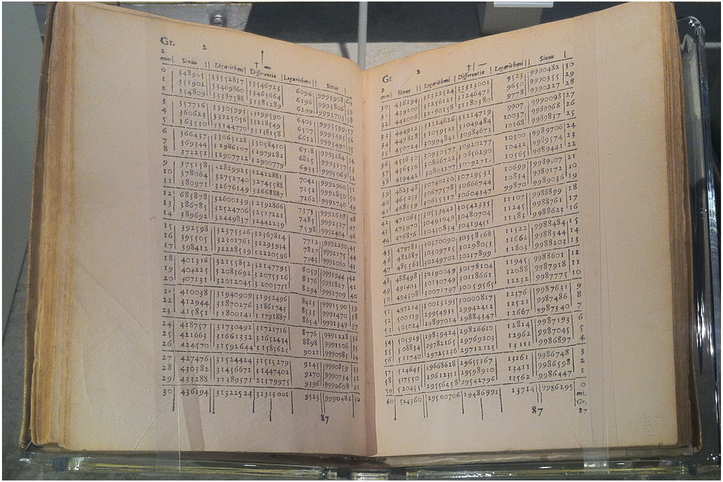

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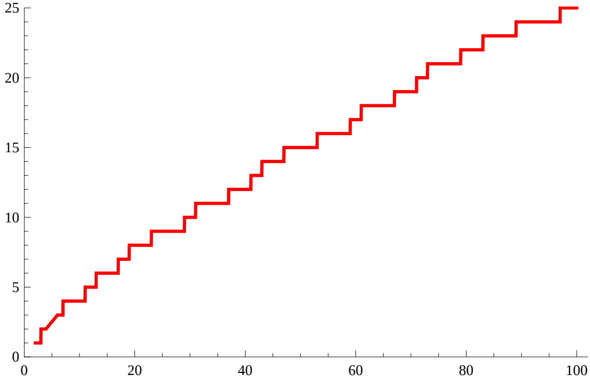

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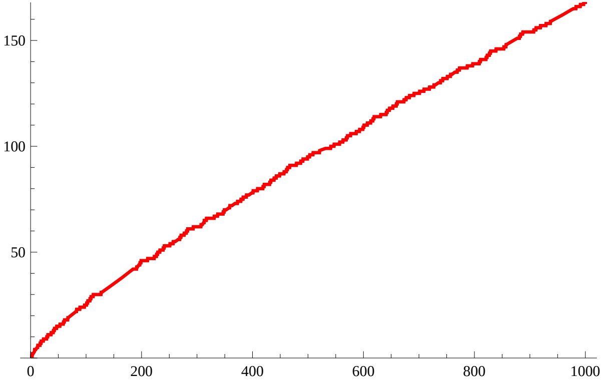

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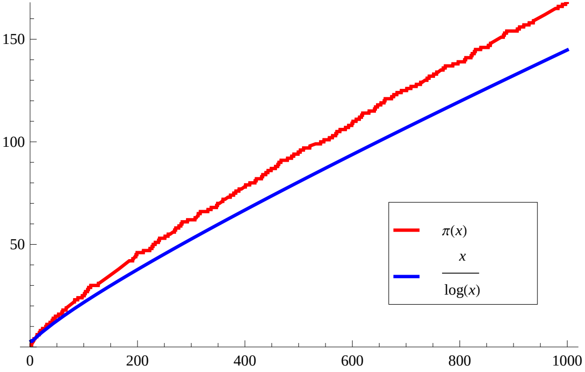

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Questions?

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Harmonic Series

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Let's go back to infinitude of primes. Euler gave the following proof:

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However, the RHS is divergent while the product is finite. This is a contradiction. □

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$$\begin{aligned} 2^{2n+1} = (1+1)^{2n+1} &= \binom{2n+1}{0} + \binom{2n+1}{1} + \ldots + \binom{2n+1}{2n+1} \\ &\geq \binom{2n+1}{n} + \binom{2n+1}{n+1} \end{aligned}$$

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since $\binom{2n+1}{n} = \binom{2n+1}{n+1}$ (exercise!).

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But $\binom{2n+1}{n} = \frac{(2n+1) \cdot 2n \dots \cdot (n+2)}{n!}$ so all of p_k, ..., p_m divide $\binom{2n+1}{n}$.

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But $\binom{2n+1}{n} = \frac{(2n+1) \cdot 2n \dots \cdot (n+2)}{n!}$ so all of p_k, ..., p_m divide $\binom{2n+1}{n}$. Thus their product is less than $\binom{2n+1}{n}$.

Proof of Lemma Continued

We had

$$n \log 4 \geq \log \binom{2n+1}{n}.$$

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$$\theta(2n+1) - \theta(n+1) \leq \log \binom{2n+1}{n} \leq n \log 4.$$

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Recall that we wish to prove that $\frac{\pi(x)\log x}{x} \leq A$.

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But $\sqrt{x}$ is much smaller than $\frac{x}{\log x}$. $\square$

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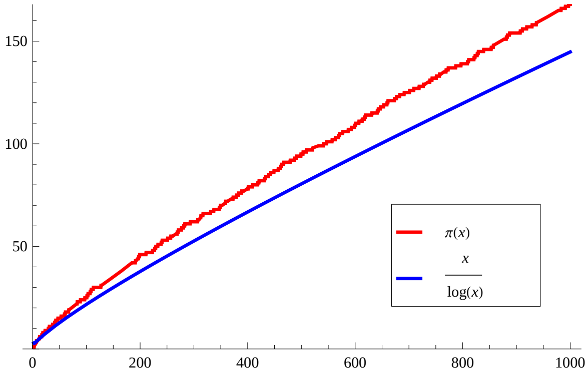
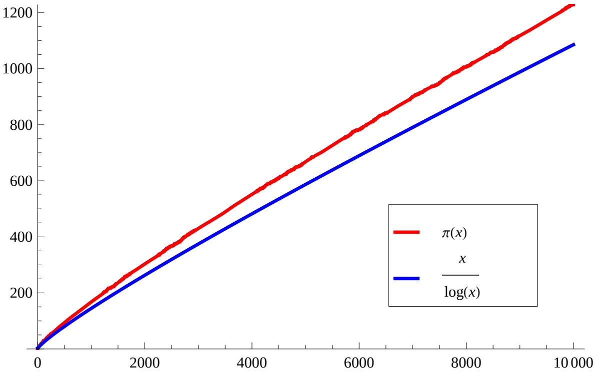
- Riemann considered this as a function of a complex variable
- PNT relies on showing that $\zeta(1 + it) \neq 0$ where $i = \sqrt{-1}$ and $t \neq 0$

Error in PNT

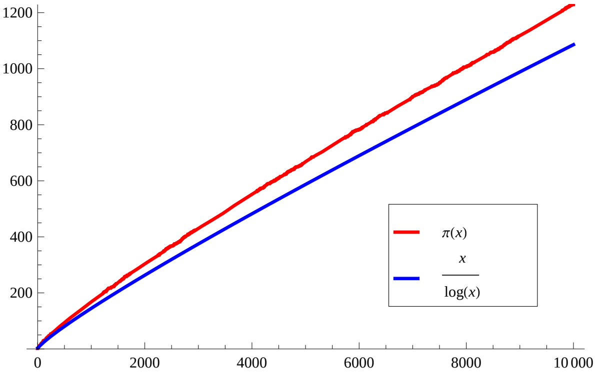


Error in PNT



1200
1000
800
600
400
200
0
2000
4000
6000
8000
10 000
$\pi(x)$
$\frac{x}{\log(x)}$

Error in PNT



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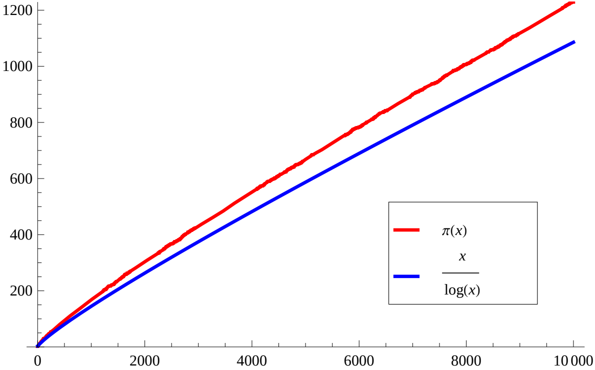

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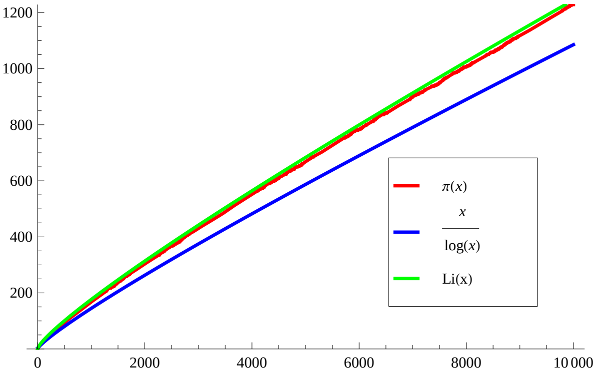

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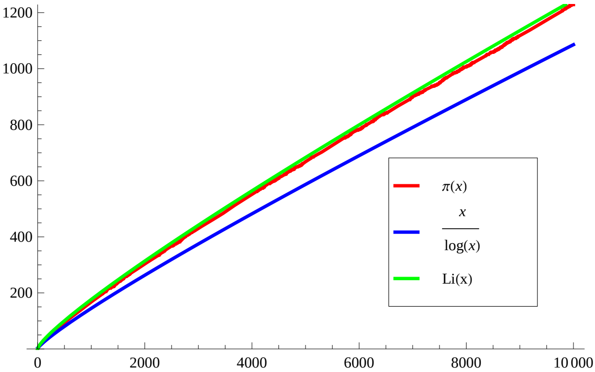

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  - PNT $\iff \lim_{x\to\infty} \frac{\pi(x)}{\mathrm{Li}(x)} = 1$

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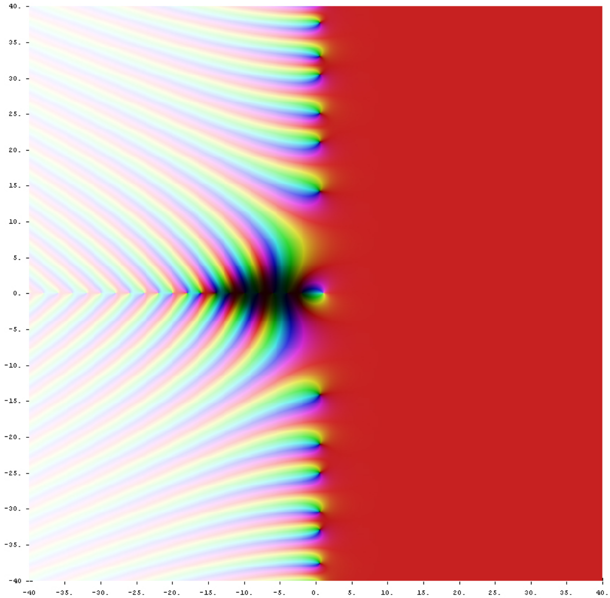

40.
35.
30.
25.
20.
15.
10.
5.
0.
-5.
-10.
-15.
-20.
-25.
-30.
-35.
-40
-40
-35.
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5.
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- Define $\pi_2(x)$ to be the number of twin primes up to x
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Gauß' Puzzle

$$1+2+\ldots+99+100$$

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Thanks!

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Thanks!

Any questions?

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