



UCL



Prime Number Theorem

Niko Laaksonen
UCL

Of Life and Death



Of Life and Death



- Cicada

Of Life and Death

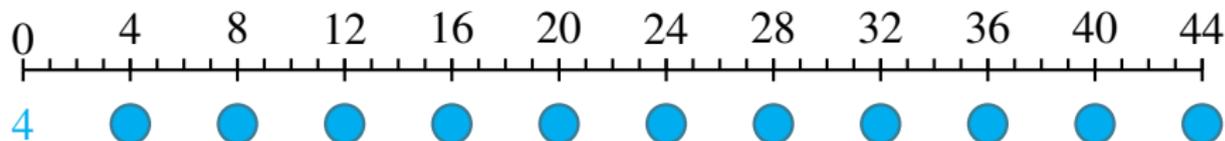


- Cicada
- Life cycles in predator–prey system

Of Life and Death



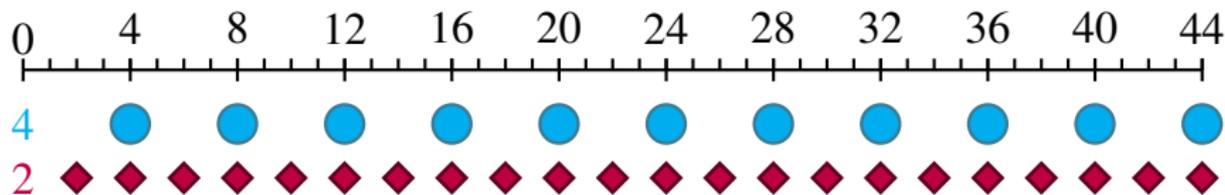
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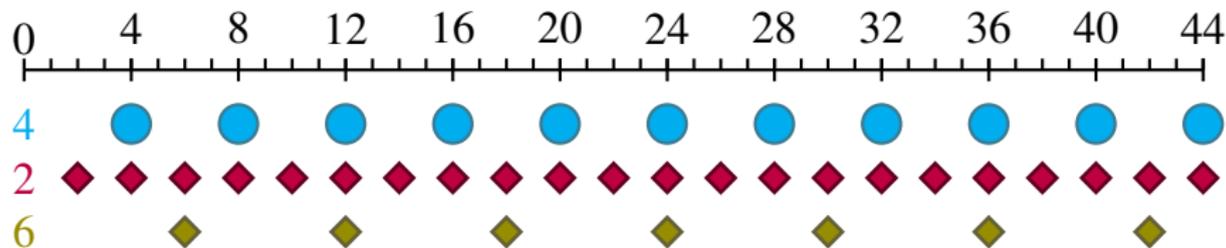
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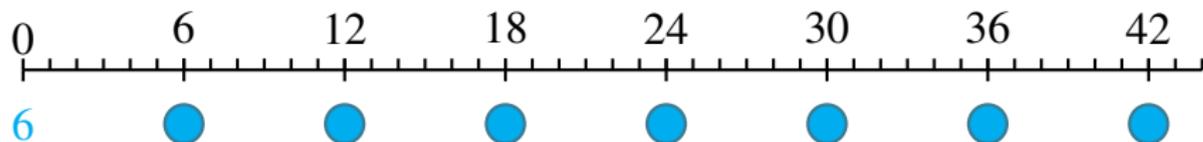
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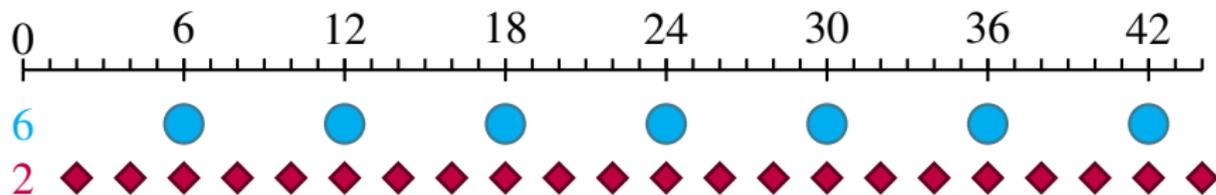
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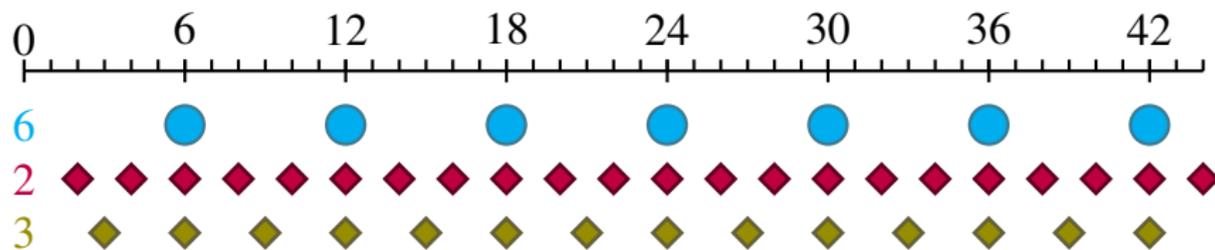
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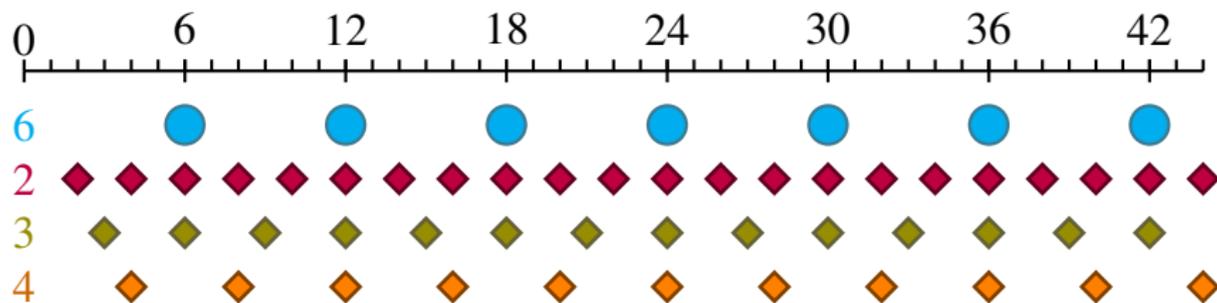
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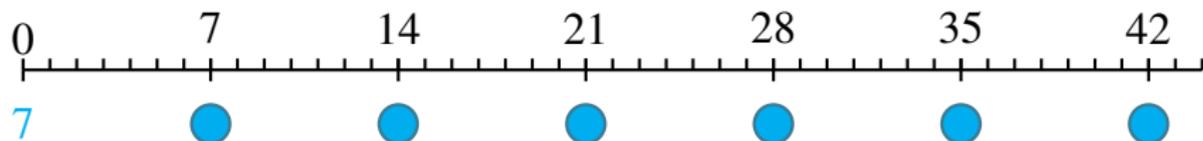
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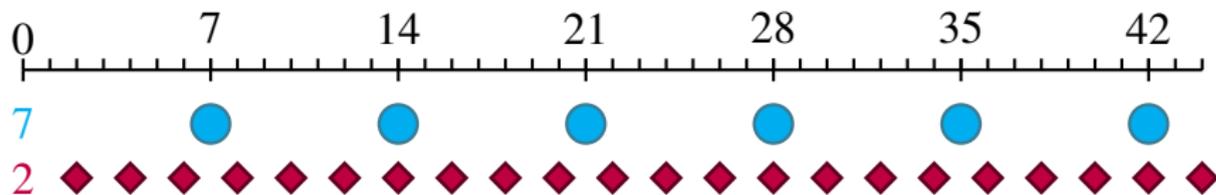
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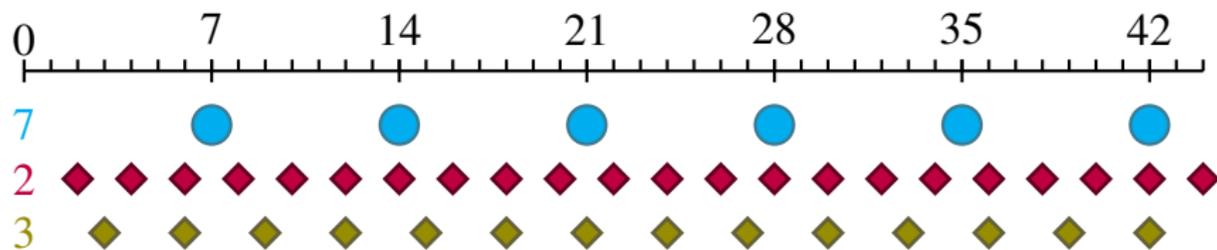
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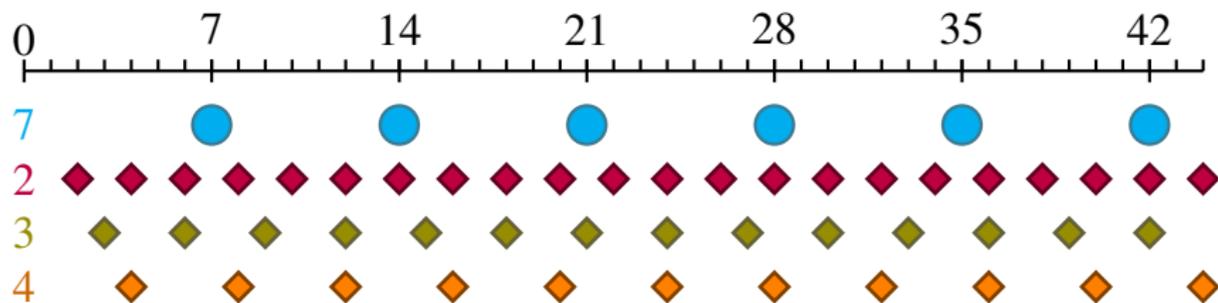
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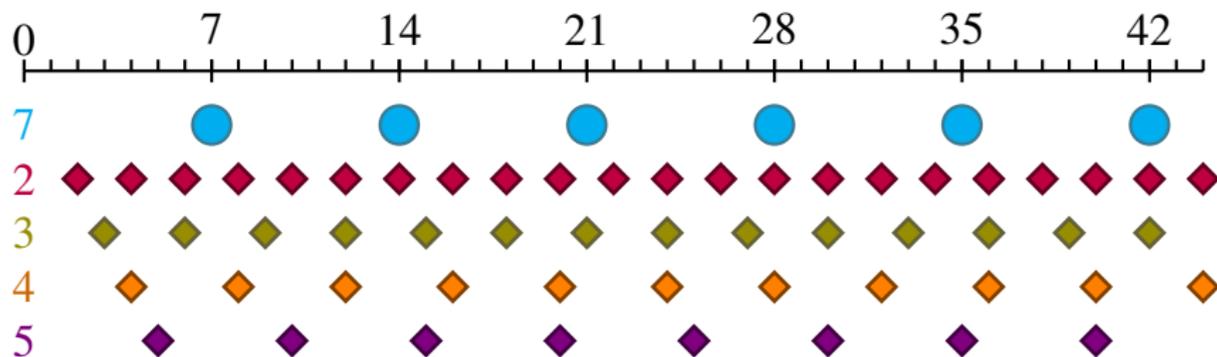
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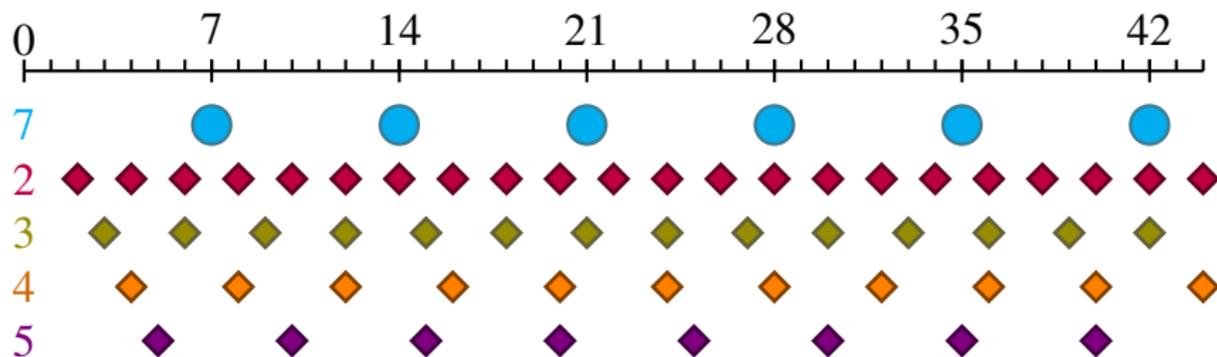
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7 has no non-trivial divisors!

Prime Number Facts

Definition

A positive number p is a **prime number** if its only positive divisors are 1 and itself.

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- 10 333? Still a prime. So is 103 333
- However,
 $1\ 033\ 333 = 7 \times 43 \times 3433$.
- In fact every number can be written as a product of primes.
 - This representation is unique.



How many primes are there?

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- Largest publicly known prime is $2^{43112609} - 1$

So how do we find them?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
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Sieve of Eratosthenes (ca. 200 BC)

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- Gaps between primes
 - How large?

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In other words, there are arbitrarily long gaps between primes.

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 - *Infinitely many*
- How large gaps are there between primes?

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Prime Number Facts Recap

- How many of them are there?
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- How large gaps are there between primes?
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- How many primes up to some number x ?

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3.0001	2

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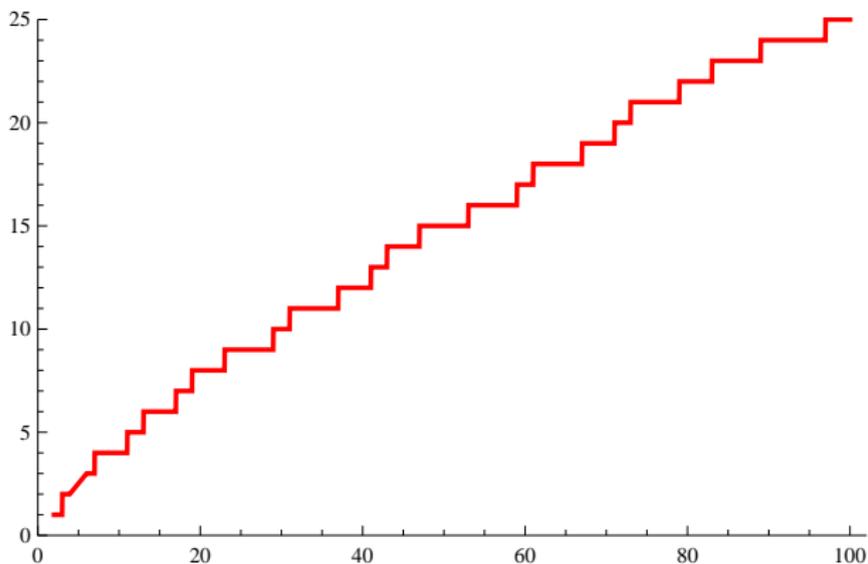
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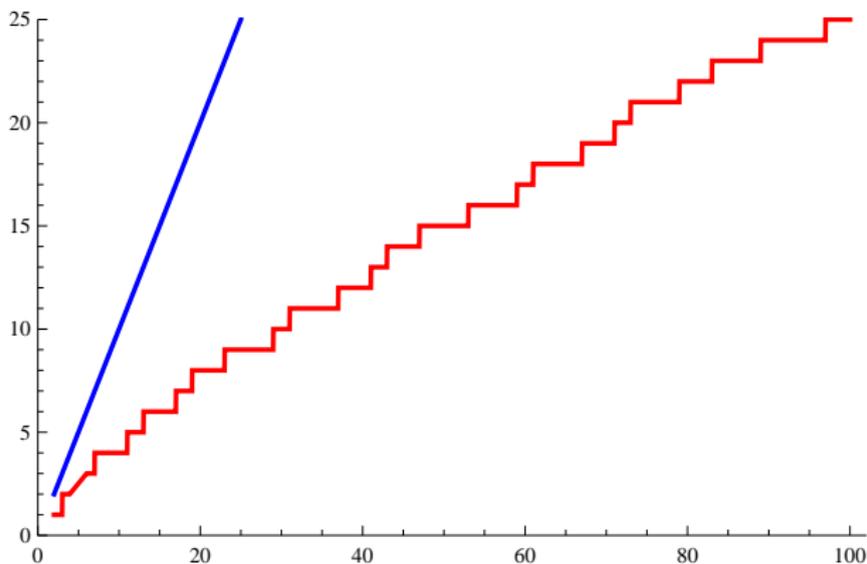
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- Can we find a formula for $\pi(x)$?
- Not a smooth function—**step function**
- Can we find $f(x)$ such that $f(n) = \pi(n)$ if n is a prime?
- i.e. Can we approximate $\pi(x)$ by a smooth function $f(x)$?

A Formula for Primes

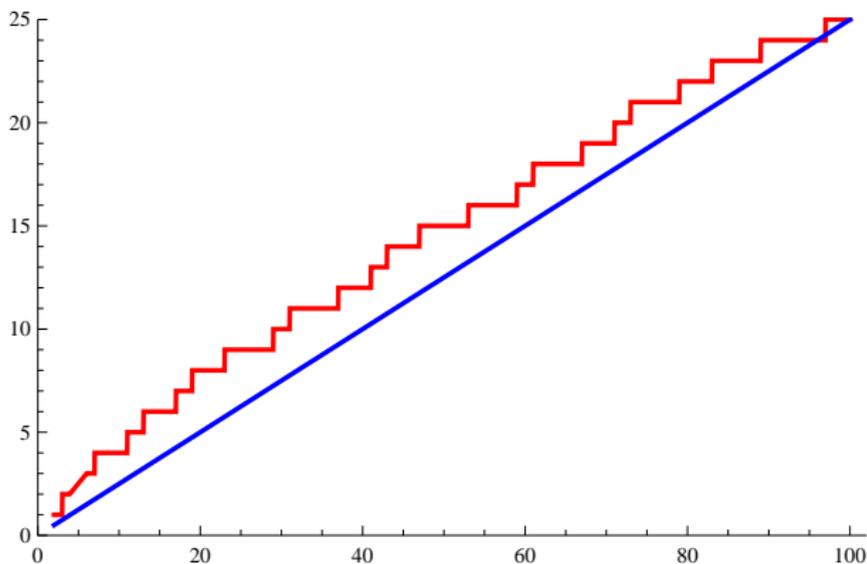


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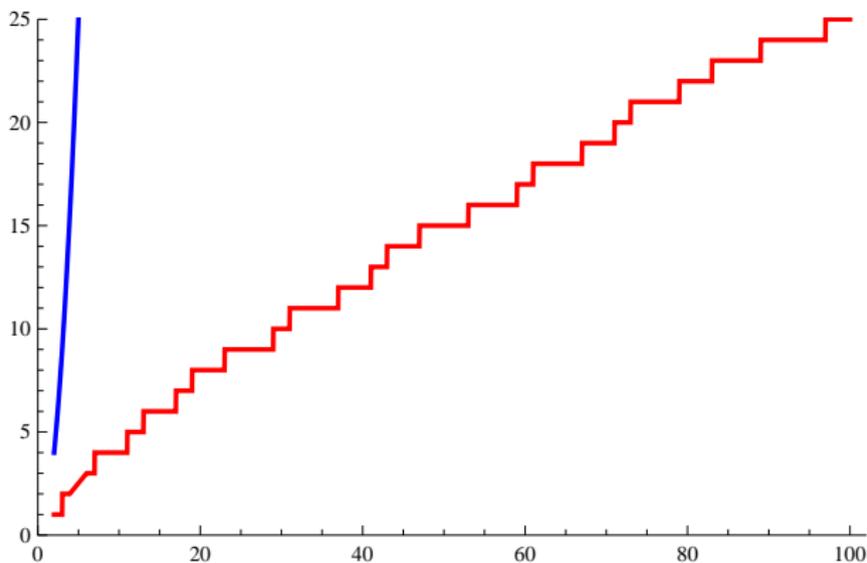
$$f(x) = x$$

A Formula for Primes



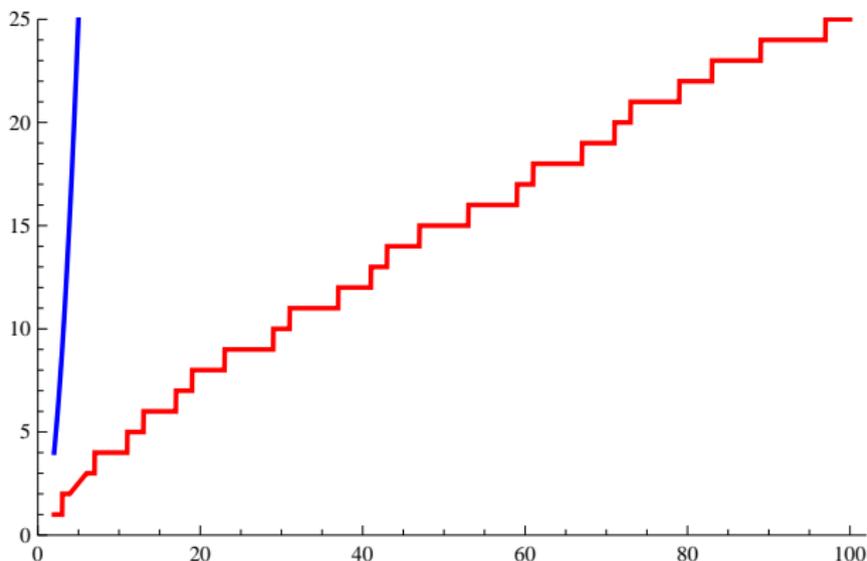
$$f(x) = \frac{x}{4}$$

A Formula for Primes



$$f(x) = x^2$$

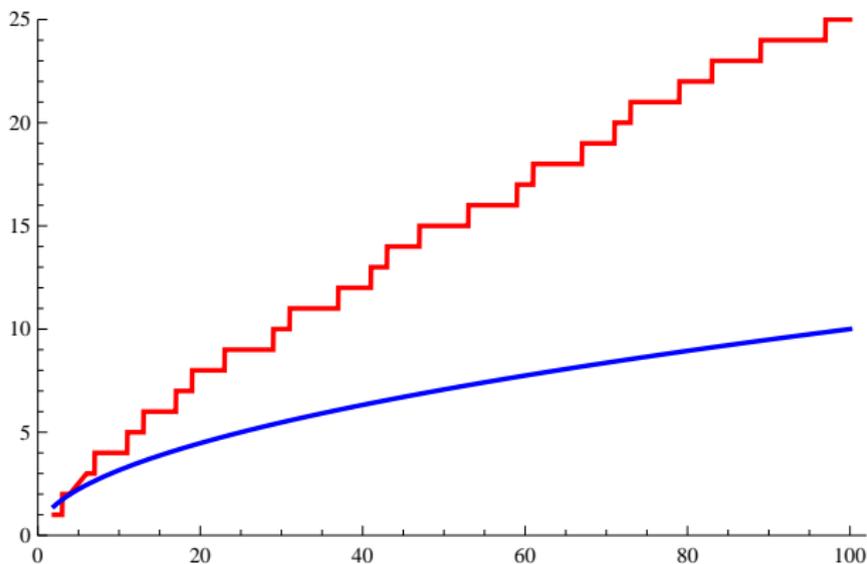
A Formula for Primes



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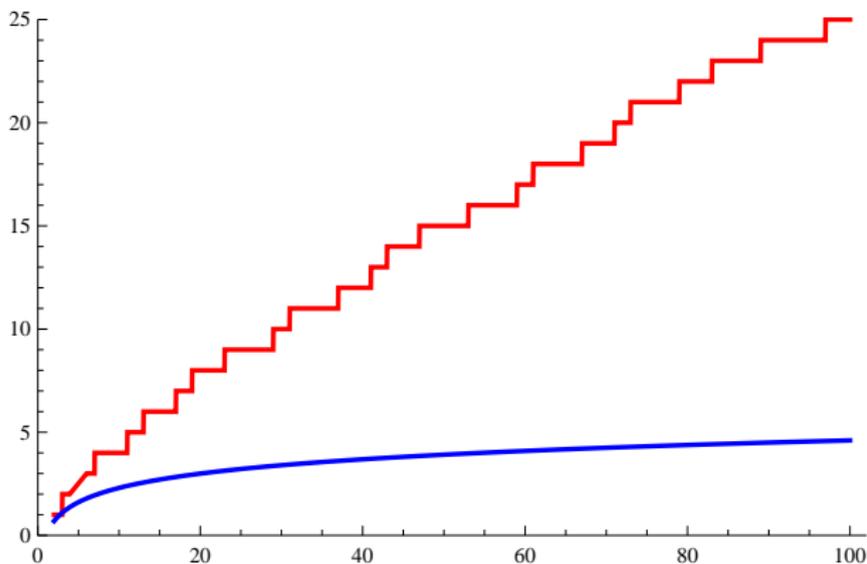
In fact, no polynomial will do!

A Formula for Primes



$$f(x) = \sqrt{x}$$

A Formula for Primes



$$f(x) = \log x$$

Gauß (1777–1855)



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- Born to a poor family

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- Born to a poor family
- Child prodigy

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- Child prodigy
 - $1 + 2 + \dots + 100 = ?$
- Didn't care about publishing his results
- Hobby: computing primes

Gauß' Answer to Our Question (at the age of 15!)

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x	$\pi(x)$	$\frac{x}{\pi(x)}$
1 000	168	5.9523
10 000	1 229	8.1367
100 000	9 592	10.4254
1 000 000	78 498	12.7392
10 000 000	664 579	15.0471
100 000 000	5 761 455	17.3567
1 000 000 000	50 847 534	19.6666

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4.4731

Gauß' Answer to Our Question (at the age of 15!)

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4.4731	10 000	1 229	8.1367
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Gauß' Way

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a	mm	Stamm	Differenz	Stamm	a	mm	Stamm	Differenz	Stamm
0	14394	1551317	1546713	6094	10	146124	1552544	1551100	9121
1	14400	1546960	1546960	6196	11	147100	1548111	1546421	9657
2	14400	1537788	1538183	6299	12	148100	1543611	1545030	9778
3	14774	1529099	1529490	6403	13	149100	1539111	15414719	9707
4	16664	1521000	1518149	6497	14	150100	1534611	1540484	10017
5	16100	1514470	1518183	6612	15	151100	1530111	15394671	10168
6	16647	1506512	1508410	6718	16	152100	1525611	15384671	10100
7	16914	1498609	1497983	6835	17	153100	1521117	15374671	10441
8	17118	1490721	1490272	6931	18	154100	1516617	15364671	10565
9	17118	1482821	1481181	7041	19	155100	1512117	15354671	10690
10	17804	1474740	1474183	7153	20	156100	1507617	15344671	10814
11	18097	1466614	1466183	7261	21	157100	1503117	15334671	10938
12	18187	1458514	1458183	7373	22	158100	1498617	15324671	11062
13	18187	1450414	1450183	7481	23	159100	1494117	15314671	11186
14	18809	1442300	1442119	7593	24	160100	1489617	15304671	11310
15	18809	1434200	1434000	7701	25	161100	1485117	15294671	11434
16	19421	1426100	1426184	7813	26	162100	1480617	15284671	11558
17	19421	1418000	1418191	7921	27	163100	1476117	15274671	11682
18	19421	1410000	1410191	8031	28	164100	1471617	15264671	11806
19	20033	1402000	1402191	8143	29	165100	1467117	15254671	11930
20	20033	1394000	1394191	8251	30	166100	1462617	15244671	12054
21	20645	1386000	1386191	8363	31	167100	1458117	15234671	12178
22	20645	1378000	1378191	8471	32	168100	1453617	15224671	12302
23	21257	1370000	1370191	8583	33	169100	1449117	15214671	12426
24	21257	1362000	1362191	8691	34	170100	1444617	15204671	12550
25	21869	1354000	1354191	8803	35	171100	1440117	15194671	12674
26	21869	1346000	1346191	8911	36	172100	1435617	15184671	12798
27	22481	1338000	1338191	9023	37	173100	1431117	15174671	12922
28	22481	1330000	1330191	9131	38	174100	1426617	15164671	13046
29	23093	1322000	1322191	9243	39	175100	1422117	15154671	13170
30	23093	1314000	1314191	9351	40	176100	1417617	15144671	13294
					41	177100	1413117	15134671	13418
					42	178100	1408617	15124671	13542
					43	179100	1404117	15114671	13666
					44	180100	1399617	15104671	13790
					45	181100	1395117	15094671	13914
					46	182100	1390617	15084671	14038
					47	183100	1386117	15074671	14162
					48	184100	1381617	15064671	14286
					49	185100	1377117	15054671	14410
					50	186100	1372617	15044671	14534
					51	187100	1368117	15034671	14658
					52	188100	1363617	15024671	14782
					53	189100	1359117	15014671	14906
					54	190100	1354617	15004671	15030
					55	191100	1350117	14994671	15154
					56	192100	1345617	14984671	15278
					57	193100	1341117	14974671	15402
					58	194100	1336617	14964671	15526
					59	195100	1332117	14954671	15650
					60	196100	1327617	14944671	15774

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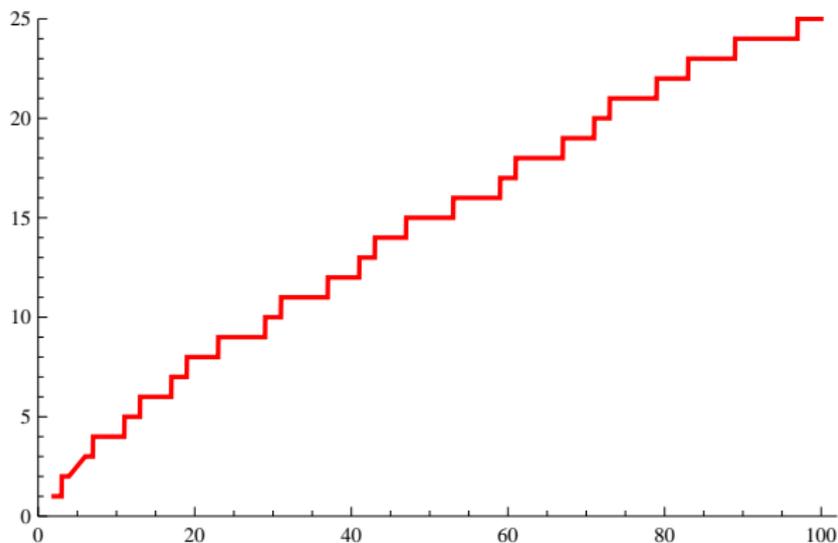
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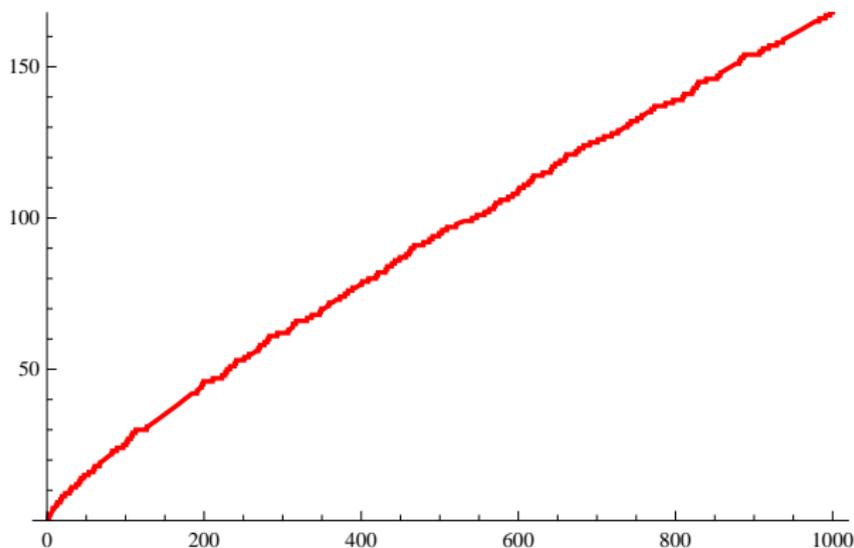


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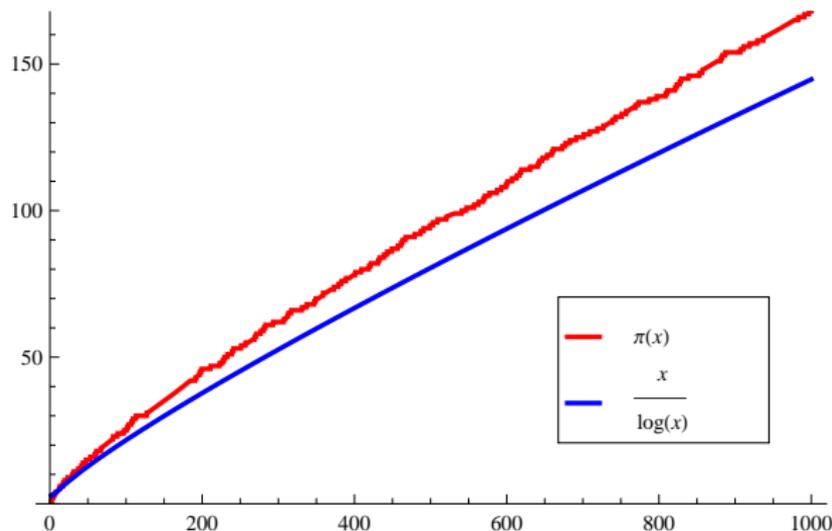


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Questions?

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$$\sum_{i=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

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However, the RHS is divergent while the product is finite. This is a contradiction. □

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where the sum is taken over all primes less than x .

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$$2^{2n+1} = (1+1)^{2n+1} = \binom{2n+1}{0} + \binom{2n+1}{1} + \dots + \binom{2n+1}{2n+1}$$

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since $\binom{2n+1}{n} = \binom{2n+1}{n+1}$ (exercise!).

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$$\theta(2n+1) - \theta(n+1) \leq \log \binom{2n+1}{n} \leq n \log 4.$$

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Recall that we wish to prove that $\frac{\pi(x)\log x}{x} \leq A$.

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$$\pi(x) \leq \frac{2x \log 4}{\log x} + \pi(\sqrt{x}) \leq \frac{2x \log 4}{\log x} + \sqrt{x}.$$

But \sqrt{x} is much smaller than $\frac{x}{\log x}$. □

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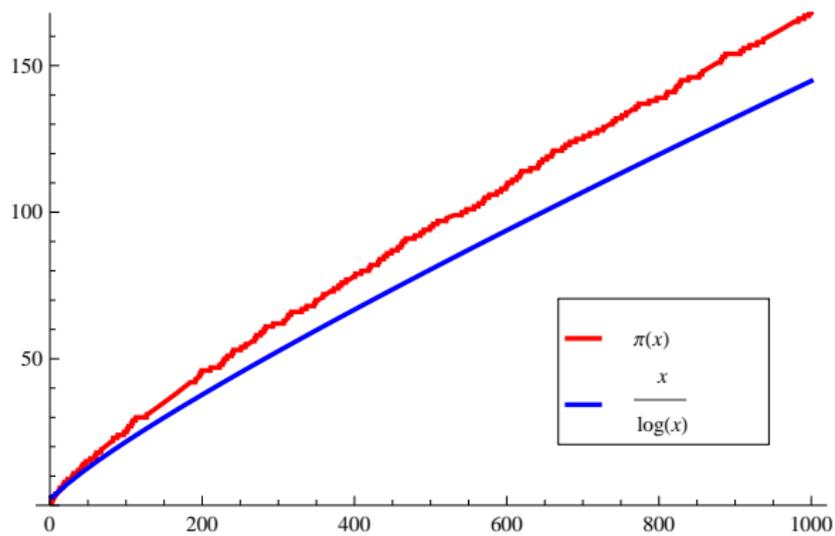
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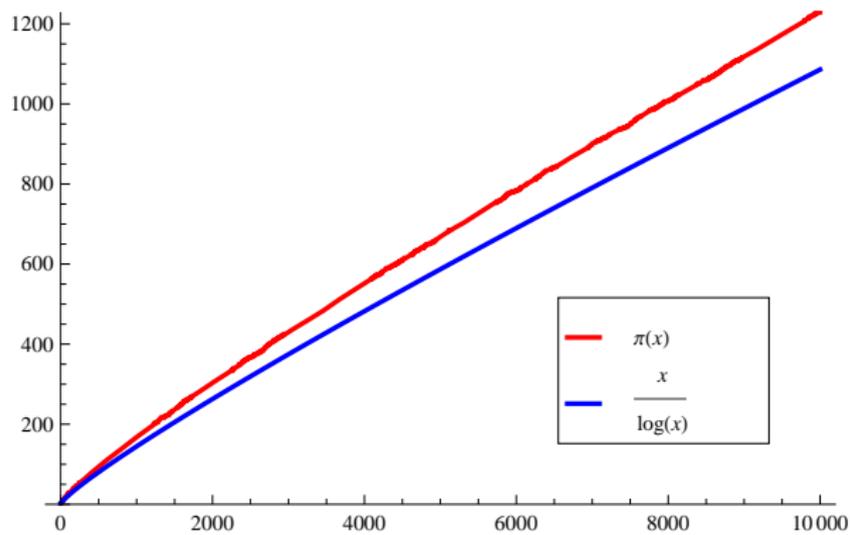
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- PNT relies on showing that $\zeta(1 + it) \neq 0$ where $i = \sqrt{-1}$ and $t \neq 0$

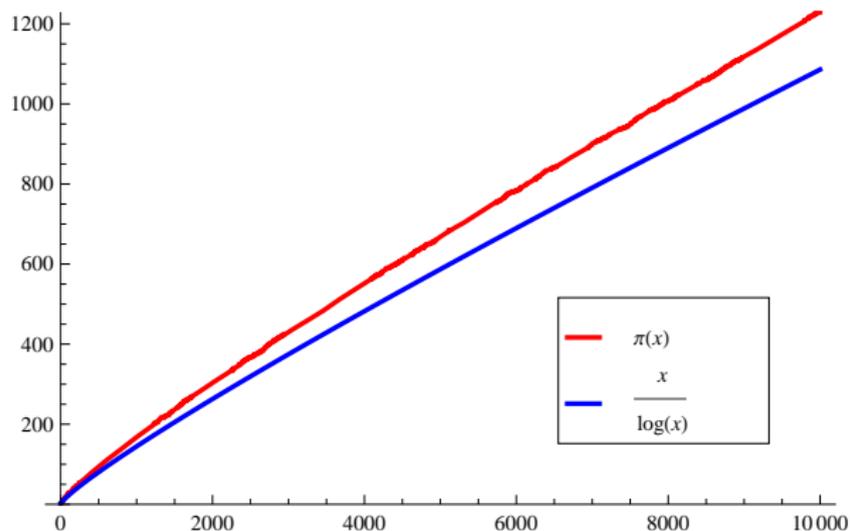
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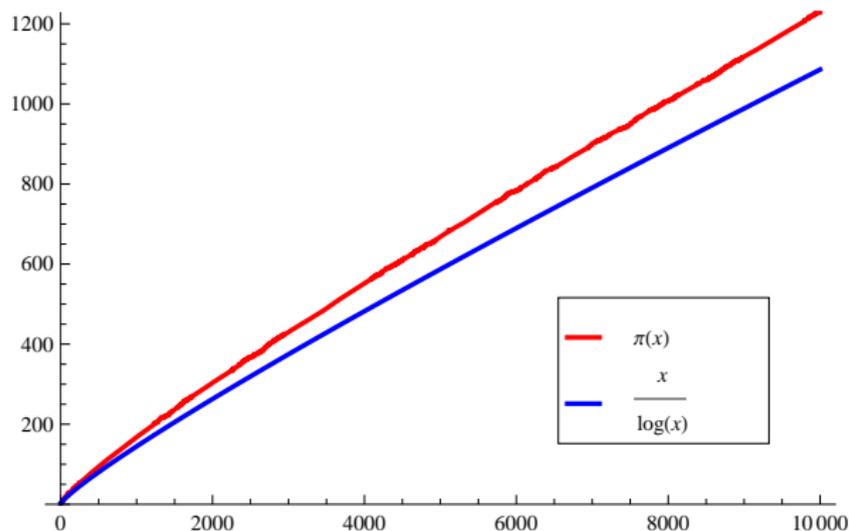


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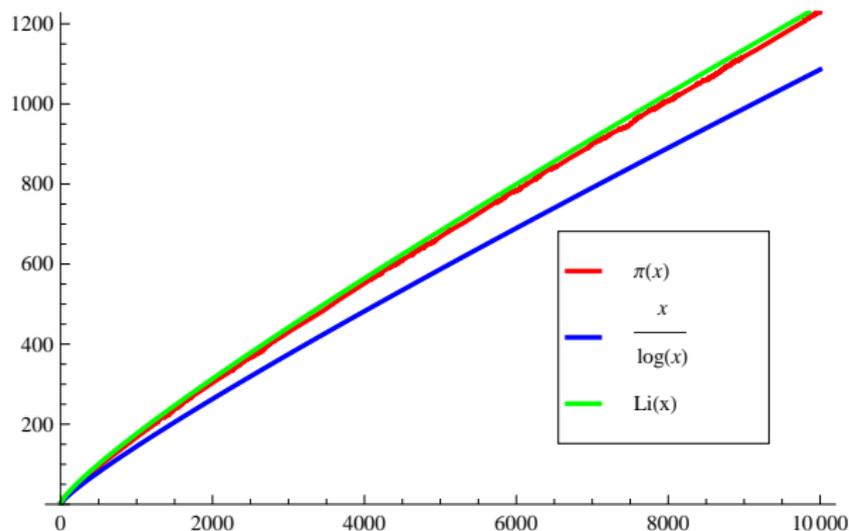
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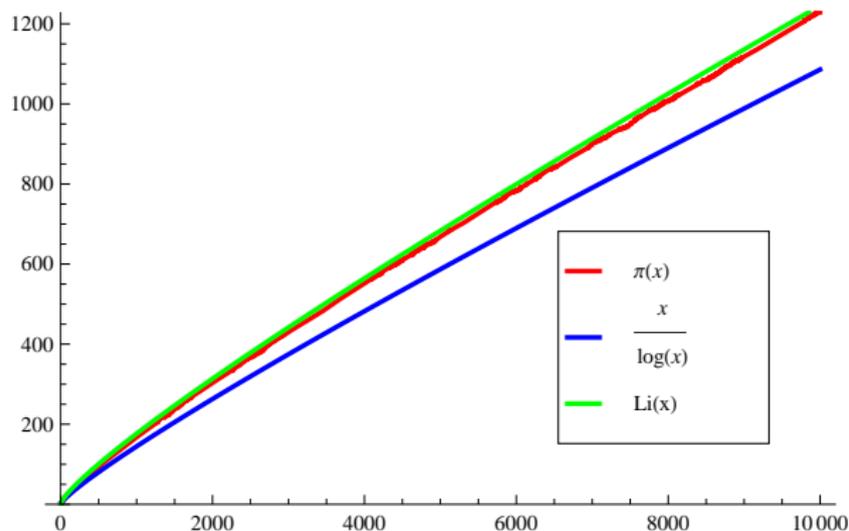
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- PNT $\iff \lim_{x \rightarrow \infty} \frac{\pi(x)}{\text{Li}(x)} = 1$

What's Left?

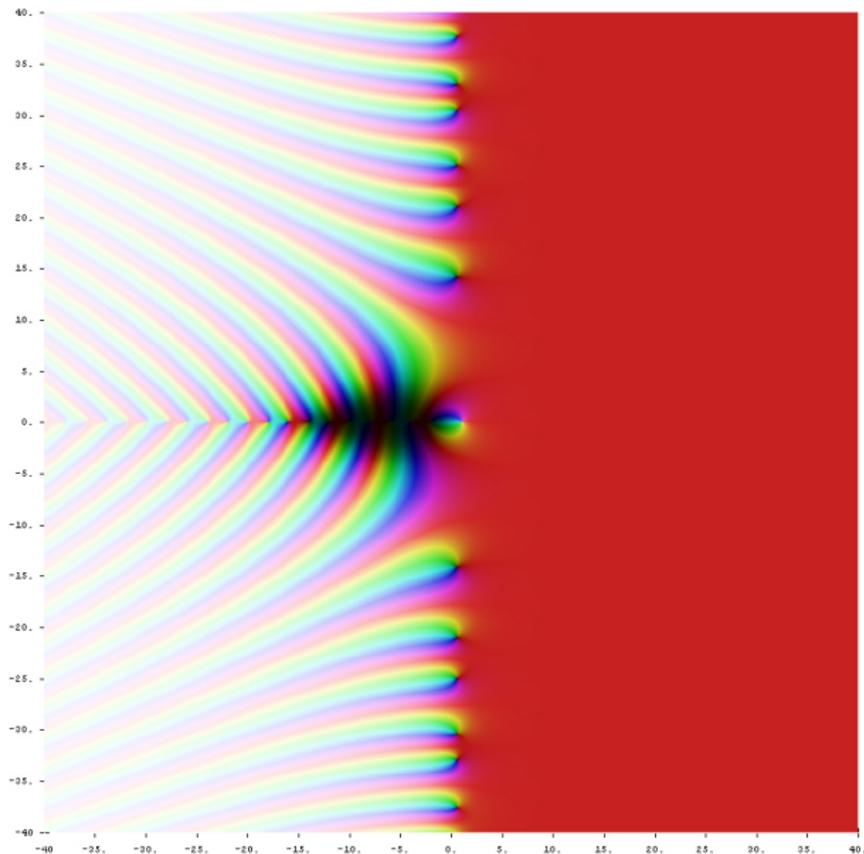
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Gauß' Puzzle

$$1 + 2 + \dots + 99 + 100$$

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$$+100 + 99 + \dots + 2 + 1$$

Gauß' Puzzle

$$\begin{aligned} &1 + 2 + \dots + 99 + 100 \\ &+ 100 + 99 + \dots + 2 + 1 \\ &= 101 + 101 + \dots 101 + 101 \end{aligned}$$

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Thanks!

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Thanks!

Any questions?

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