Newman's Short Proof of the Prime Number Theorem

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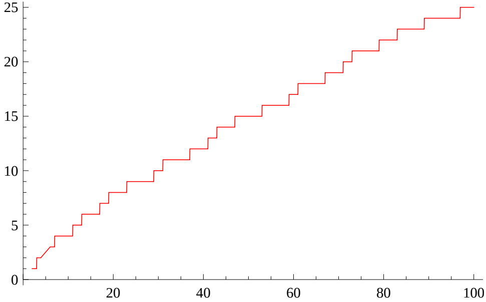

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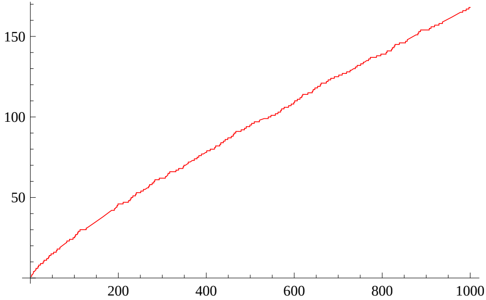

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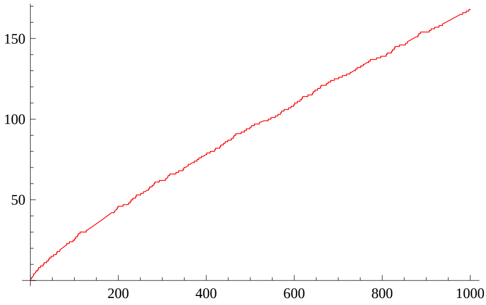

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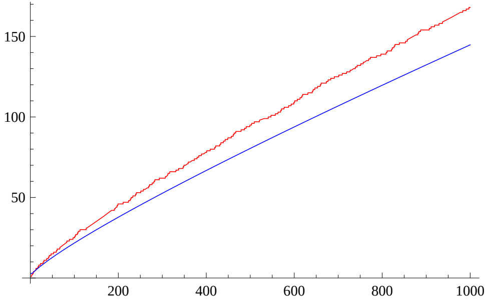

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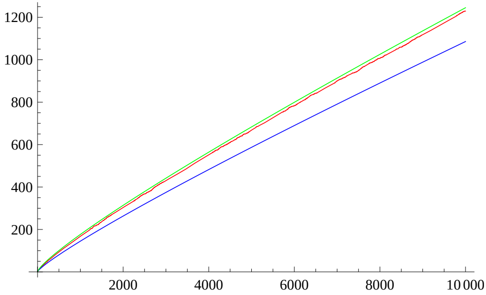

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Apply Analytic Theorem with $f(t) = \theta(e^t)e^{-t} - 1$, and $g(z) = \frac{\Phi(z+1)}{z+1} - \frac{1}{z}$. □

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Suppose that for some $\lambda > 1$ there are large x with $\theta(x) \geq \lambda x$. For such x:

$$\int_x^{\lambda x} \frac{\theta(t) - t}{t^2}\, dt \geq \int_x^{\lambda x} \frac{\lambda x - t}{t^2}\, dt$$

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This is a contradiction. Similarly for $\lambda < 1$. $\square$

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Thus

$$(1-\epsilon)^{-1} \frac{\theta(x)}{x} \geq \frac{\pi(x) \log x}{x} \geq \frac{\theta(x)}{x}.$$

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Proof of Analytic Theorem - I

Analytic Theorem

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$$\lim_{T\to\infty} g_T(0) = g(0).$$

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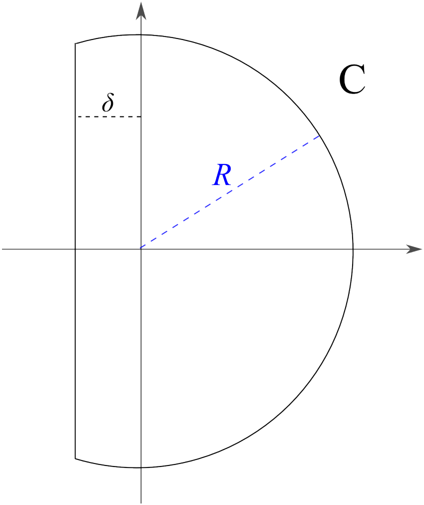

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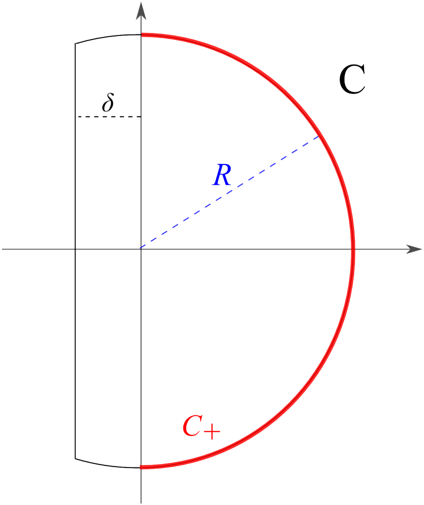

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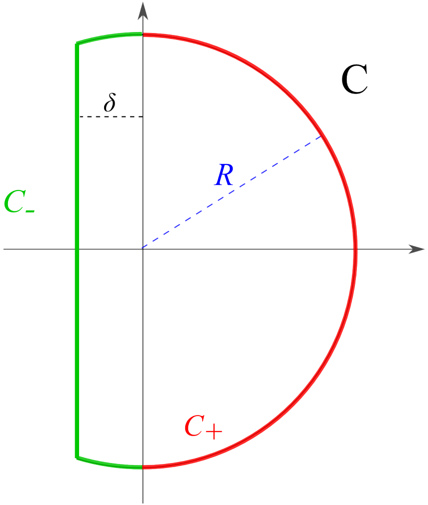

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$$g(0) - g_T(0) = \frac{1}{2\pi i} \int_C (g(z) - g_T(z)) e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z}.$$

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Hence we can estimate the integral on C_+ by

$$\frac{1}{2\pi} \cdot R\pi \cdot B \cdot \frac{2}{R^2} = \frac{B}{R}.$$

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The remaining integral is $\displaystyle\int_{C_-} g(z)\left(1+\frac{z^2}{R^2}\right)e^{zT}\frac{dz}{z}$.

This tends to 0 as $T \to \infty$, because $e^{zT} \to 0$ rapidly for $\Re(z) < 0$. Hence

$$\limsup_{T\to\infty} |g(0) - g_T(0)| \leq \frac{2B}{R}.$$

Letting $R \to \infty$ proves the theorem. $\square$

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  - Write $2^{2n} = \binom{2n}{0} + \ldots + \binom{2n}{2n} \geq \binom{2n}{n} \geq \prod_{n<p\leq 2n} = e^{\theta(2n)-\theta(n)}$

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- $\zeta(s) \neq 0$ and $\Phi(s) - 1/(s-1)$ is holomorphic for $\Re(s) \geq 1$.
  - Primes in arithmetic progressions

References

Zagier, D., *Newman's Short Proof of the Prime Number Theorem*, *The American Mathematical Monthly*, Vol. 104, No. 8 (Oct., 1997)