Nevanlinna Theory

On Meromorphic Functions and Their Value Distribution Theory

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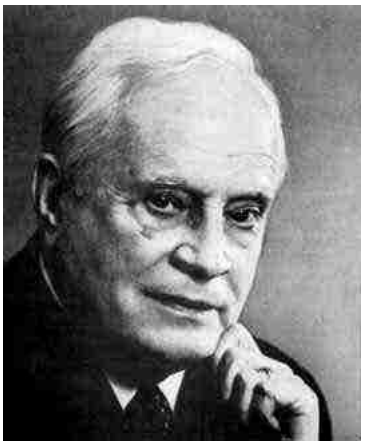
Value distribution theory is the study of how often a complex function assumes different values. By the Fundamental Theorem of Algebra we know that any polynomial over $\mathbb{C}$ is completely determined by its zeros up to a constant multiple. But what can we say about general entire functions? How many zeros must, or can, these functions have? How many times can two meromorphic functions assume the same value on the same set? How does Nevanlinna fit into all this? If you wish to hear the answers to these questions come and join us on the 26th of October.



"Rolf Nevanlinna, creator of the modern theory of meromorphic functions" -Hayman, W.K.

With this talk I'm hoping to introduce some concepts that were just out of reach in the first complex analysis course at UCL. Eventually my goal is to naturally end up at a point where, in order to make further progress, it is essential to introduce the theory of Nevanlinna. No previous knowledge is assumed from the audience, but some familiarity with basic complex analysis can be helpful.

1 Polynomials

Value distribution theory tries to answer the question of how many times does a certain function take different complex values, and what is the distribution — or density — of these points. We begin by looking at the simple case of a polynomial $f\colon \mathbb{C} \to \mathbb{C}$ of degree n. This well serve to introduce the ideas of the theory and to suggest parallels to a wider class of functions.

By the Fundamental Theorem of Algebra f has exactly n zeros counted with multiplicities. In other words, the function f takes the value 0 exactly n times. On the other hand, we may ask how many times must f take any other complex value, say, $a \in \mathbb{C}$.

In the case of polynomials this is easy to see. Consider the polynomial $f - a$ which is also of degree n. Then this polynomial takes the value 0 exactly n times. Thus f takes the value a exactly n times (with multiplicities). This is clearly not true for a general real polynomial!

There are two things that this observation allows us to deduce. Letting $a_1, \ldots, a_n$ be the zeroes of f, we can write $f(z) = c(z - a_1) \ldots (z - a_n)$ for some $c \in \mathbb{C}$. The crux of this is that *any* two complex polynomials with the same roots can only differ by a constant multiple.

We also have that

$$\lim_{|z| \to \infty} \frac{f(z)}{z^n} = c.$$

More precisely, by the *Maximum modulus principle* we know that the maximum value attained by f on a disc of radius R centred at the origin lies on the boundary. So we obtain the following statement about the growth of f.

$$\sup_{|z| \leq R} |f(z)| = O(R^n) \quad \text{as } R \to \infty.$$

The converse is true as well. That is, if the maximum modulus of a function grows as r^n then it is a polynomial.

We can see the interplay of three distinct concepts here: the distribution of the zeros of a function, the growth of such functions, and to what extent is the function determined by these zeros.

2 Entire Functions

Recall from Complex Analysis, `MATH2101`, that an entire function is an analytic function f on all of the complex plane. From the same course you might remember a result for the minimal growth condition for entire functions, namely, *Liouville's Theorem*. This says that any bounded entire function is a constant. Again, in light of the maximum modulus principle, it is enough to require that the maximum modulus of a function on a circle doesn't grow bigger as we move away from the origin.

The obvious problem, when trying to generalise the Fundamental Theorem of Algebra, comes from the fact that entire functions can have infinitely many zeros making the naive product of linear factors divergent. In fact, entire functions can take any value infinitely many times. We can be even more precise than this. The following is a theorem by a French mathematician **Charles Picard** given in 1879.

Picard's Theorem *Any non-constant entire function can omit at most one finite value, and takes on every other complex value infinitely many times.*

The original proof of this uses modular functions, but we will later see how it actually follows as a simple consequence of Nevanlinna theory.

Let us now look at the specific example $f(z) = e^z$. We know that f is an entire function, and that it never takes the value zero. Hence, by Picard's theorem f must attain every other complex value infinitely many times. This presents another problem to us: a non-constant entire function may not have any zeros at all! Thus it makes sense to suspect that two entire functions with the same zeroes can not only differ by a constant, but also by a multiple of some sort of exponential factor. A very explicit formulation was given by **Weierstraß** in

1876, which assumes a form similar to the Fundamental Theorem of Algebra.

Theorem *Let $\{a_n\}$ be a sequence of nonzero complex numbers such that* $\lim|a_n| = \infty$. *Then any entire function which vanishes exactly at each a_n and has a zero of order m at the origin is of the form*

$$f(z) = e^{g(z)} z^m \prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right),$$

where g is entire and the ***canonical factors*** *E_n are defined by*

$$E_k(z) = (1-z)e^{z+z^2/2+\ldots+z^k/k}.$$

The reason for introducing these canonical factors is to make our infinite product converge. It remains to investigate the growth of such functions in relation to the distribution of zeros. As was said before the exponential factor we had to introduce gives us a little trouble. It can arbitrarily affect the rate of growth of f without introducing any additional zeros. It turns out that if we limit ourselves to entire functions, which maximum modulus doesn't grow too fast, we can ensure that g is a polynomial. The following definition makes this distinction more precise.

Definition Let f be an entire function. Then the **order of growth of** f is given by

$$\rho_f = \inf\{\rho > 0 : |f(z)| \leq Ae^{B|z|^\rho}\},$$

where $A, B > 0$ are constants.

Jacques Hadamard showed that for a function of order ρ the degree of the polynomial in the exponent is at most ρ. Moreover, if we assume that $f(0) \neq 0$ then

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta - \sum_{k=1}^{n} \log\left|\frac{R}{a_k}\right|, \tag{1}$$

where $a_1, \ldots, a_n$ are the zeros of f inside $|z| < R$. Equation (1) is called **Jensen's formula**. It allows us to give an estimate for the distribution of zeros of f. For this, we need to note

that

$$\begin{aligned}\sum_{k=1}^{n} \log\left|\frac{R}{a_k}\right| &= \sum_{k=1}^{n} \int_{|a_k|}^{R} \frac{1}{t}\, dt \\ &= \sum_{k=1}^{n} \int_{0}^{R} \chi_k(t) \frac{1}{t}\, dt \\ &= \int_{0}^{R} \frac{1}{t} \sum_{k=1}^{n} \chi_k(t)\, dt \\ &= \int_{0}^{R} \frac{n(t,0)}{t}\, dt,\end{aligned}$$

where $n(r,0)$ is the number of zeros of f inside the open disc of radius r centred at the origin, and $\chi_k(r)$ is $1 -$ (the characteristic function of the disk $r \leq |a_k|$). Substituting this into Jensen's Formula yields

$$\int_0^r n(t,0)\frac{dt}{t} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta - \log|f(0)|. \tag{2}$$

In particular if f is of finite order of growth the above formula allows us to give an asymptotic upper bound for the number of zeros: $n(r,0) = O(r^\rho)$. This was improved upon by Borel who showed that

$$\limsup_{r\to\infty} \frac{\log n(r,a)}{\log r} = \rho$$

except at most for one value $a \in \mathbb{C}$ for which the earlier inequality holds instead (think about e^z for example).

So we have seen that even though entire functions aren't as easy to deal with as polynomials it is still possible to overcome these difficulties. Moreover, Borel's result shows that all entire functions of finite order have their value distribution completely determined by their order, and that the distribution is same for each value $a \in \mathbb{C}$ with at most one exception. We shall now shift our focus to meromorphic functions, which will also be the starting point of Nevanlinna theory.

3 Meromorphic Functions

Definition Let f be a function on an open set $\Omega \subseteq \mathbb{C}$. We say that f is **meromorphic** if there exists a sequence of points $\{z_n\}_{n=0}^{\infty}$ in $\mathbb{C}$ such that $\{z_n\}$ has no limit points in Ω, and

(i) f is analytic in $\Omega \setminus \{z_0, z_1, \ldots\}$, and

(ii) f has poles at the points z_0, z_1, z_2, ...

It is possible to show that any meromorphic function is the quotient of two entire functions, which is sometimes taken as the definition.

Now things start getting more complicated. It no longer makes sense to measure the growth of a function by calculating the maximum modulus, since a meromorphic function can attain the value ∞ on a disc of finite radius. We can say something about these functions, though. For example, Picard's Theorem generalises easily. Let f be a meromorphic function which omits a finite value a. Then $\frac{1}{f-a}$ is an entire function, which by Picard's Theorem can omit at most one value. Hence any meromorphic function can omit at most two finite values.

Moreover, since we can write any meromorphic function as a quotient of two entire functions, one might be lead to suspect that it is reasonable to attempt to measure the growth of f by studying the growth of the numerator and the denominator separately. This was the idea Borel had, and he managed to generalise some of the earlier results to meromorphic functions by defining the order of such f to be the maximum of the order of the denominator and the numerator. It however became apparent that any further development of these ideas would be difficult from this point of view. (Lehto, 1982)

4 Nevanlinna Theory

It was time for **Rolf Nevanlinna** to revolutionise the study of meromorphic functions. He did this through a series of publications in 1922–1925 at the age of 26. His key idea was to use Jensen's Formula with a slight modification.

Define the real function $\log^+(x) = \max(\log x, 0)$. Nevanlinna then used this function to construct three real-valued functions to measure the behaviour of f. In view of (1) he defined

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta.$$

This is called the **(mean) proximity function** which essentially measures how big f is on the circle $|z| = r$. Furthermore, denote by $n(t, f)$ the number of poles of f in $|z| \leq t$, counting multiplicities, and let us assume that $f(0) \neq \infty$ (we are still assuming that $f(0) \neq 0$, too). Define

$$N(r, f) = \int_0^r \frac{n(t, f)}{t}\, dt,$$

which is Nevanlinna's **counting function**. It clearly counts the logarithmic average of the poles of f inside the disk $|z| < r$.

We now rewrite Jensen's Formula in terms of $\log^+$. Notice that for $x > 0$ it is true that $\log x = \log^+ x - \log^+ \frac{1}{x}$. Thus

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(Re^{i\theta})|\, d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(Re^{i\theta})|}\, d\theta \\ &= m(R, f) - m\left(R, \frac{1}{f}\right).\end{aligned}$$

In fact, for Jensen's formula to hold for meromorphic functions we need to include a similar sum to that of the one over zeros to the equation, but taken over the poles instead. Anyway, with this in mind, and recalling how we converted the sum in question into an integral, we can finally rewrite Formula (1) as

$$\log|f(0)| = m(R, f) - m\left(R, \frac{1}{f}\right) - N\left(R, \frac{1}{f}\right) + N(R, f). \tag{3}$$

First of all, in the spirit of the discussion so far, while $m(R, f)$ measures the mean size of f on the circle, $m\left(R, \frac{1}{f}\right)$ looks at the same aspect from another point of view by measuring how close f is to 0 on the circle on average. Similarly for $N(r, f)$ and $N\left(R, \frac{1}{f}\right)$. The important thing to notice here is that the left-hand side is constant even when $R \to \infty$. Nevanlinna realised that this would allow him to effectively measure the growth of f. He defined the **(Nevanlinna) characteristic function**

$$T(r, f) = m(r, f) + N(r, f),$$

which can be used to rewrite Equation (3) in a very compact form

$$T\left(R, \frac{1}{f}\right) = T(R, f) - \log|f(0)|. \tag{4}$$

This realisation has been described by Ahlfors (1976), a student of Nevanlinna, as the point

when Nevanlinna theory was born.

The function T is sometimes — originally by Nevanlinna as well — referred to as the **affinity of f for ∞**, or in other words "a measure of how much f likes ∞" (substitute ∞ with 0 for $\frac{1}{f}$, or with a for $\frac{1}{f-a}$). The reason for this terminology becomes apparent from the deduction Nevanlinna made from Equation (4). This is what he referred to as the **The First Main Theorem**

$$T\left(R, \frac{1}{f}\right) = T(R, f) + O(1), \tag{5}$$

where $O(1)$ is bounded. This is really just a restatement of (4), because $\log|f(0)|$ is a finite constant. The implications, however, are huge. What Theorem (5) says is that f likes the value ∞ as much as 0. It is actually possible to generalise this for any $a \in \mathbb{C}$. Moreover, if the value of the characteristic function is asymptotically unbounded (which is the case with all interesting functions) then if f has many zeros — that is, $N(r, 1/f)$ grows fast — then f has to spend a lot of time away from the value 0 as $m(r, 1/f)$ must be small. This is because the characteristic function must behave in essentially the same way for all values $a \in \mathbb{C}$ by the First Main Theorem. Let us look at a concrete example to illustrate this point.

Again, let $f(z) = e^z$. We can now say much more about f than what was possible with Picard's theorem. As said before, f never attains the value 0 nor the value ∞ as it is entire. Thus $N(r, e^z) = N(r, e^{-z}) = 0\ \forall r$, and by the argument above $m(r, e^z)$ and $m(r, e^{-z})$ have to compensate. And this is indeed true! If you recall the familiar graph of the real exponential function you will notice how it is very close to 0 to the left of the y-axis and grows very quickly to the right. Now all we need to do is to notice that the real exponential function gives the magnitude of the complex exponential, because $|e^z| = e^{\mathrm{Re}\ z}$, and we can deduce the same result for all $z \in \mathbb{C}$. Hence, in fact, f is close to 0 on the whole left half-plane, and close to ∞ on the right one exactly as predicted by Nevanlinna theory!

Having gone through all the trouble of developing the theory for meromorphic functions, it would be a shame not to give an example. So here we go. Possibly the simplest class of meromorphic functions is the rational functions, that is, the functions which are quotients of two complex polynomials. Let $f(z) = \frac{P(z)}{Q(z)}$, where $P(0) \neq 0 \neq Q(0)$, and $\deg P = n$, $\deg Q = m$. We first consider the case $m \geq n$. Then if we pick a disc of radius r_0 containing the pole furthest away from origin (which we can do since all the poles are at finite points,

and there is a finite amount of them on all of $\mathbb{C}$), we have $n(r, f) = m$ for $r \geq r_0$. Hence

$$\begin{aligned} N(r, f) &= \int_0^{r_0} \frac{n(t, f)}{t}\, dt + \int_{r_0}^{r} \frac{m}{t}\, dt \\ &= m(\log r - \log r_0) + O(1) \\ &= m \log r + O(1) \end{aligned}$$

as the first integral is clearly bounded, and $n(0, f) = 0$. Recall from Section 1 that the growth of a polynomial is dominated by the highest-order term. Hence for any $\epsilon > 0$ there exists an $r_0 > 0$ such that for all $r = |z| > r_0$ we have

$$(1 - \epsilon)|a_n|r^n \leq |P(z)| \leq (1 + \epsilon)|a_n|r^n,$$

where a_n is the leading coefficient of P. So for $r \geq r_0$ we can write $|P(z)| = |a_n|r^n(1 + o(1))$ and $|Q(z)| = |b_m|r^m(1 + o(1))$, where $g = o(h)$ means $\lim_{x\to\infty} \frac{g(x)}{h(x)} = 0$. But then

$$\log|f(z)| = \log|a_n| + n \log r + \log|1 + o(1)| - \log|b_m| - m \log r - \log|1 + o(1)|,$$

and as $m \geq n$ we find that $\log^+|f| = O(1)$ and it follows that $m(r, f) = O(1)$. We can now calculate the characteristic function of f, which clearly is

$$T(r, f) = m \log r + O(1) = O(\log r).$$

In the case $m < n$ we can no longer control the growth of f in the same manner, but by the First Main Theorem and the previous case we get

$$T(r, f) = T\left(r, \frac{1}{f}\right) + O(1) = n \log r + O(1) = O(\log r).$$

Thus if f is a rational function it satisfies the growth condition $T(r, f) = O(\log r)$. Interestingly, the converse is true as well. That is, any meromorphic function f which grows as $T(r, f) = O(\log r)$ is a rational function. So whereas the polynomials are characterised by "polynomial growth" of the maximum modulus, the rational functions are exactly the meromorphic functions with logarithmic growth (in the sense of Nevanlinna's characteristic)!

Having successfully devised a way to measure the value distribution of meromorphic functions, Nevanlinna looked more closely into his characteristic function. He wanted to understand the intrinsic relation between the terms m and N in the definition of T. It

turned out that, in general, m is much smaller than N. In the end this idea took the following form.

Second Main Theorem *Let f be a meromorphic function on $\mathbb{C}$ and let $a_1, \ldots, a_q$ be distinct finite or infinite numbers with $q > 2$. Then*

$$(q-2)T(r,f) \leq \sum_{i=1}^{q} N(r,a_i) + \text{ small terms.}$$

An important special case is given by $q = 3$, which is where Nevanlinna arrived first as well. We then have

$$T(r,f) \leq N(r,a) + N(r,b) + N(r,c) + \text{ small terms.}$$

Let us now give a short proof of Picard's Theorem in the case of meromorphic functions. Assume for contradiction that a meromorphic function f omits at least three distinct values a, b, and c. Then $N(r,a) = N(r,b) = N(r,c) = 0$ for all r, which is a contradiction since T is unbounded.

5 Applications

The main unsolved problem in pure Nevanlinna theory seems to be estimating the error term in the Second Main Theorem. Not only is it an interesting problem on it's own, mathematicians (Vojta, Lang, Osgood, etc.) realised during the 20th century that good estimates of the error term have consequences in other subjects. In general, the true power of Nevanlinna theory becomes clear when one looks at the multitude of its applications. Here I would like to mention a few of those, which both are still subject to ongoing research.

Let us say that two meromorphic functions f and g share a value $a \in \mathbb{C}$ if $\{z : f(z) = a\} = \{z : g(z) = a\}$. It is then possible to prove using Nevanlinna theory that if two meromorphic functions f and g share five distinct values then either $f \equiv g$ or they are both constant. Notice that, for example, e^z and e^{-z} share the values 0, 1, -1, and ∞. Thus the number 5 in the theorem is strict.Charak (2009) lists sources related to this topic.

Another application, which goes much deeper, is the connection between Nevanlinna theory and Diophantine approximation. Paul Vojta expressed in great detail the analogy between Nevanlinna theory and Diophantine approximation in 1987. He showed that many of the functions of Nevanlinna theory have corresponding functions or quantities in Diophantine approximation, and moreover that there existed theorems analogous to the First and Second Main Theorems. If you wish to know more about this it is suggested to have a look at Vojta's notes on the subject given in the references as well as Cherry and Ye (2001).

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