MATH222 — Calculus 3

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This course is based on the book Stewart, *Calculus: Multivariable Calculus*, Eighth Edition, CENGAGE Learning.

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Chapter 1

Sequences & Series

1.1 Sequences

Recommended problems from §11.1: **31, 32, 40, 46, 52, 62, 63, 79, 80**.

Definition 1.1. A **sequence** is an infinite ordered list (set) of numbers.

For example, the sequence of non-negative even numbers would be 0, 2, 4, 6, . . . In general we use subscript notation for sequences so an arbitrary sequence can be written as $a_1, a_2, a_3, \ldots, a_n, \ldots$ or more concisely as $\{a_n\}_{n=1}^{\infty}$ (if it's clear from the context, we sometimes just write $\{a_n\}$ or even a_n by abusing the notation).

Example. Consider $\{b_n\}_{n=1}^{\infty}$ with $b_n = \frac{1}{n}$. This gives the following sequence of terms: 1, $\frac{1}{2}$, $\frac{1}{3}$, ...

Notice that the terms in the sequence b_n get arbitrarily close to 0 (although they never reach it) as n increases. We say that the **limit** of the sequence b_n is 0 (or that b_n **converges** to 0). In this section our focus of study is to decide whether a sequence converges and to determine its limit. We state the idea of a limit more formally in the next definition[1].

Definition 1.2. A sequence $\{a_n\}$ **converges** to a **limit** a if for every $\varepsilon > 0$ there is an $N \in \mathbb{R}$ such that

$$n > N \implies |a_n - a| < \varepsilon.$$

(the above line can be read as '*if* $n > N$ *holds, then* $|a_n - a| < \varepsilon$ *must hold too*'). For such a $\{a_n\}$ we write $\lim_{n\to\infty} a_n = a$, or $a_n \to a$ as $n \to \infty$.

Notice that in this definition N depends on ε so we should think of N as a function of ε (and sometimes we even emphasise this by writing $N(\varepsilon)$). We can rephrase the definition of the limit in terms of Figure 1.1:

[1]The downside of this definition is that in order to prove that a sequence converges we need to know *a priori* what the limit is. It is possible to define the convergence of sequences without mentioning limits, but that is beyond the scope of this course.

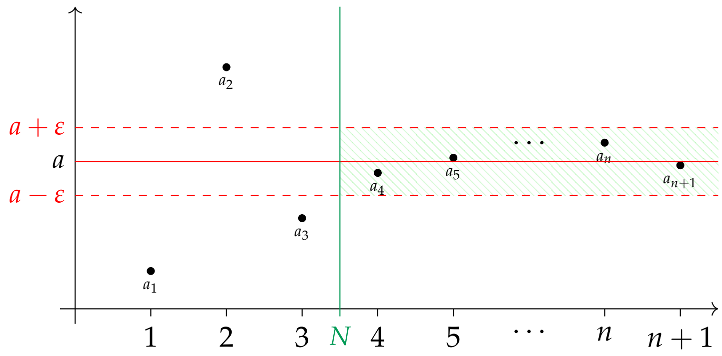


Figure 1.1: Illustration of the definition of a limit.

(i) first you give me a fixed $\varepsilon > 0$ and construct the ε-tube (in red),

(ii) then I need to be able to provide you an index N such that if I draw a vertical line at N (in green) then all the following terms in the sequence lie inside the ε-tube.

It is possible to prove (although we won't do it here) that the limit, *if it exists*, is unique. In practice, it is fairly tedious to use Definition 1.2 and we'll develop better tools for determining the convergence of a sequence and for finding its limit. For now, let's do an example to see how to use the definition.

Example. Consider the sequence defined by $a_n = \frac{1}{n^2+1}$. We make an educated guess that this sequence should converge to 0. Let's prove it by using Definition 1.2. To do this we first fix an $\varepsilon > 0$. We then need to find an N (depending on ε) such that

$$n > N \implies \left| \frac{1}{n^2+1} - 0 \right| < \varepsilon. \tag{1.1}$$

We solve the last inequality for n:

$$\begin{aligned} \left| \frac{1}{n^2+1} \right| &< \varepsilon \\ \frac{1}{n^2+1} &< \varepsilon && \text{(since } 1/(n^2+1) \text{ is positive)} \\ \frac{1}{\varepsilon} &< n^2+1 && \text{(by cross-multiplication)} \\ \frac{1}{\varepsilon} - 1 &< n^2. \end{aligned}$$

One might be tempted to just set N equal to $\sqrt{\varepsilon^{-1} - 1}$, but this would not always work (if $\varepsilon > 1$ (why?)). So we have to be a bit more careful (even if in reality we are more interested in the *small* ε). However, notice that if $\varepsilon \geq 1$ then (1.1) is trivially satisfied (because $\frac{1}{n^2+1} < 1 \leq \varepsilon$ for any $n \neq 0$) so we can choose any N, say, $N = 1$. Hence we get that by letting

$$N = \begin{cases} \sqrt{\frac{1}{\varepsilon} - 1}, & \text{if } \varepsilon < 1, \\ 1, & \text{otherwise,} \end{cases}$$

the implication (1.1) is satisfied.

Not all sequences converge: if a sequence does not have a limit then we say that it **diverges**. For example, consider the sequence given by

$$c_n = \sqrt{n}. \tag{1.2}$$

The terms in this sequence keep increasing indefinitely so it is impossible for them to approach a fixed number. More precisely we say:

Definition 1.3. A sequence $\{a_n\}$ diverges to ∞ if for every $M \in \mathbb{R}$ there is $N \in \mathbb{R}$ such that

$$n > N \implies a_n > M.$$

For such sequences we write $\lim_{n\to\infty} a_n = \infty$, or $a_n \to \infty$ as $n \to \infty$.

Exercise. Prove that the sequence (1.2) diverges to ∞.

We have an analogous definition for diverging to $-\infty$ (i.e. for any m we need an N such that $n > N \implies a_n < m$).

It is natural to ask whether there are divergent sequences that do not diverge to $\pm\infty$. The answer is yes, consider for example the sequence given by $a_n = (-1)^n$, then the terms of this sequence keep oscillating between 1 and -1,

$$1, -1, 1, -1, 1, -1, \ldots,$$

so they never approach a single value and as such the sequence diverges. This illustrates an important point (which we'll return to later) that a *convergent sequence is necessarily bounded* (i.e. if a_n converges then there are m and M such that $m < a_n < M$ for all n), but the *converse is not necessarily true* (i.e. not all bounded sequence converge, such as the one given above). We'll see in the next lecture what extra condition is required to guarantee convergence. Notice that it is even possible to have a divergent unbounded sequence that doesn't diverge to either ∞ or $-\infty$, e.g. $b_n = (-1)^n n$.

For now, let's return back to computing limits of sequences. It is imperative to learn to do this *fast* and *efficiently*. Our fundamental tool will be the next theorem.

Theorem 1.4 (Algebra of Limits). *Suppose we have two convergent sequences $\{a_n\}$ and $\{b_n\}$ such that $a_n \to a$ and $b_n \to b$, and let $c \in \mathbb{R}$ be a constant. Then*

(i) $a_n + b_n \to a + b$,

(ii) $ca_n \to ca$,

(iii) $a_n b_n \to ab$,

(iv) *suppose* $b \neq 0$ *(so eventually* $b_n \neq 0$*), then* $\dfrac{a_n}{b_n} \to \dfrac{a}{b}$,

(v) *if* $a_n > 0$ *and* $p > 0$*, then* $a_n^p \to a^p$,

as $n \to \infty$.

It is important to note that we can only apply the above theorem if we know that *both* of the sequences a_n and b_n converge individually (where appropriate). That is, it's not possible to apply this theorem directly in the case when both a_n and b_n are divergent (we'll see an example of this later). Notice that we can deduce various other identities by applying the above rules: e.g. $a_n - b_n \to a - b$ follows from rules (i) and (ii), (cf. p. 737 in the book).

Example. Let $a_n = \dfrac{2n^2+1}{3n^2-n+3}$. To find the limit of a_n as $n \to \infty$, we use Algebra of Limits (AoL) and the fact that $\lim_{n\to\infty} \frac{1}{n} = 0$ (prove it!). We use the trick of dividing the numerator and the denominator by the largest power of n.

$$\begin{aligned}
\lim_{n\to\infty} \frac{2n^2+1}{3n^2-n+3} &= \lim_{n\to\infty} \frac{2+\frac{1}{n^2}}{3-\frac{1}{n}+\frac{3}{n^2}} \\
&= \frac{\lim_{n\to\infty}(2+\frac{1}{n^2})}{\lim_{n\to\infty}(3-\frac{1}{n}+\frac{3}{n^2})} && \text{(by AoL (iv))} \\
&= \frac{\lim_{n\to\infty} 2 + \lim_{n\to\infty} \frac{1}{n^2}}{\lim_{n\to\infty} 3 - \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} \frac{3}{n^2}} && \text{(by AoL (i) and (ii))} \\
&= \frac{2+0}{3-0+0} \\
&\text{(by AoL (iii) and since the limit of a constant sequence is itself))} \\
&= \frac{2}{3}.
\end{aligned}$$

Another useful tool for determining limits is the following theorem, which says that if a sequence is bounded from above and below by sequences converging to the same limit, then the original sequence itself is also convergent and converges to the same limit. In other words,

Theorem 1.5 (Sandwich/Squeeze Theorem)**.** *Suppose we have two sequences a_n and c_n with $a_n \to a$, $c_n \to a$. Also assume for large enough*[2] *n (i.e. for all $n > n_0$ for some fixed n_0) we have that*

$$a_n \le b_n \le c_n.$$

Then it follows that $b_n \to a$ as $n \to \infty$.

It also immediately follows that,

Corollary 1.6. *Let $\{a_n\}$ be a sequence such that $|a_n| \to 0$. Then $a_n \to 0$.*

Proof. Left as an exercise (Hint: $-|a_n| \le a_n \le |a_n|$). □

Example. Let $a_n = \frac{\sin n}{n^2}$. First, since $|\sin x| \le 1$, we have that

$$\frac{-1}{n^2} \le a_n \le \frac{1}{n^2},$$

for all $n \ge 1$. We already saw that $1/n \to 0$, so by AoL (iii) we get $1/n^2 \to 0$. Again, by AoL (ii) we also get that $-1/n^2 \to 0$. It thus follows from the Sandwich theorem that $a_n \to 0$.

Often we have sequences inside functions (i.e. composed with functions). If the function is continuous at the limit of the sequence, we can then pass the limit inside the function.

[2] In general for properties related to the convergence and limits of sequences the finite part (terms up to some fixed index, n_0) do not matter. In other words we can alter a finite number of terms of a sequence without changing its limit. This means that only the *tail* (terms after some fixed index) matters.

Theorem 1.7. *Suppose $a_n \to a$ and let f be a function continuous at a. Then*

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) \qquad (= f(a)).$$

Remark. When considering the limit of a sequence it's sufficient to consider the limit of the corresponding function (as long as it exists!). For example, to compute $\lim_{n\to\infty} \frac{\sin n}{n}$ (n runs over integers) it's enough to consider $\lim_{x\to\infty} \frac{\sin x}{x}$ (x runs over real numbers).

Example. Let $a_n = \sqrt[n]{4^n + 7^n} = (4^n + 7^n)^{1/n}$. First notice that

$$\sqrt[n]{7^n} \le \sqrt[n]{4^n + 7^n} \le \sqrt[n]{2 \cdot 7^n}. \tag{1.3}$$

The lower bound follows from just dropping the 4^n term (since it is positive). On the other hand, for the upper bound we replace 4^n by 7^n since $4^n \le 7^n$ for all positive n. Thus (1.3) simplifies to

$$7 \le a_n \le 2^{1/n} \cdot 7. \tag{1.4}$$

We now need to find what is $\lim_{n\to\infty} 2^{1/n}$. As 2^x is a continuous function at 0, we can apply Theorem 1.7. This yields

$$\lim_{n\to\infty} 2^{\frac{1}{n}} = 2^{\lim_{n\to\infty} \frac{1}{n}} = 2^0 = 1.$$

Using this along with (1.4) and the Sandwich theorem proves that $a_n \to 7$.

Example. Another quick example: $\sin \frac{1}{n} \to 0$ as $n \to \infty$ since $\sin x$ is continuous at $x = 0$ and $\sin 0 = 0$.

We want to prove that $\sqrt[n]{n} \to 1$, for this we will use L'Hôpital's Rule[3].

L'Hôpital's Rule. *Suppose $-\infty \leq \alpha \leq \infty$ (such an α is called an* extended real number*). If $\lim_{x\to\alpha} \frac{f(x)}{g(x)}$ is of type $0/0$, ∞/∞ or $-\infty/-\infty$ (e.g. type $0/0$ means $\lim_{x\to\alpha} f(x) = 0$ and $\lim_{x\to\alpha} g(x) = 0$) then*

$$\lim_{x\to\alpha} \frac{f(x)}{g(x)} = \lim_{x\to\alpha} \frac{f'(x)}{g'(x)},$$

provided that the limit on the right-hand side exists.

Example. Let $a_n = \sqrt[n]{n}$. Notice that

$$a_n = n^{\frac{1}{n}} = e^{\frac{1}{n}\ln n}.$$

Let's investigate the limit in the exponent. Recall that it is enough to consider the limit of the corresponding functions. Since both $x \to \infty$ and $\ln x \to \infty$, it is of the type ∞/∞, so we can apply L'Hôpital's rule:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

Since e^x is continuous at 0, it follows that $a_n \to 1$.

Example. An important example is the following:

$$\lim_{n\to\infty} r^n = \begin{cases} 0, & \text{if } |r| < 1, \\ 1, & \text{if } r = 1, \\ \text{diverges}, & \text{if } |r| > 1. \end{cases} \tag{1.5}$$

We will use this repeatedly in the future.

Example. Let's return to the familiar example of $a_n = \sqrt[n]{4^n + 7^n}$. It is also possible to find the limit of this sequence by using L'Hôpital's Rule. First we again notice that $a_n = \exp(\frac{1}{n}\ln(4^n + 7^n))$. Then,

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln(4^x + 7^x)}{x} &= \lim_{x\to\infty} \frac{\frac{(\ln 4)4^x + (\ln 7)7^x}{4^x + 7^x}}{1} && \text{(by L'Hôpital's Rule)} \\
&= \lim_{x\to\infty} \frac{(\ln 4)(\frac{4}{7})^x + \ln 7}{(\frac{4}{7})^x + 1} \\
&= \frac{0 + \ln 7}{0 + 1} = \ln 7 && \text{(by the previous example)}
\end{aligned}$$

and it follows that $a_n \to e^{\ln 7} = 7$ since e^x is continuous at $\ln 7$.

So far we've talked about proving convergence of sequences by making an educated guess for the limit and using the definition of a limit, or by using e.g. the Algebra of Limits to deconstruct the sequence into parts that we know converge. Sometimes this might not be so easy (or even possible), but there is an important subclass of sequences for which we can always guarantee convergence. First we need two definitions.

[3] It is also possible to prove it via elementary methods, but that is more complicated.

Definition 1.8. A sequence $\{a_n\}$ is called

(i) **increasing** if $a_n \leq a_{n+1}$ for all n,

(ii) **decreasing** if $a_n \geq a_{n+1}$ for all n,

(iii) **strictly increasing** if $a_n < a_{n+1}$ for all n,

(iv) **strictly decreasing** if $a_n > a_{n+1}$ for all n.

We say $\{a_n\}$ is (resp. **strictly**) **monotone** if it is either (resp. strictly) increasing or decreasing.

Example. Let $a_n = \frac{n^2+1}{n+2}$. Show that a_n is strictly increasing. We need $a_{n+1} > a_n$ for all n. That is,

$$\begin{aligned} \frac{(n+1)^2+1}{(n+1)+2} &> \frac{n^2+1}{n+2} \\ (n^2+2n+2)(n+2) &> (n^2+1)(n+3) \\ n^2+5n+1 &> 0 \end{aligned}$$

which holds since $n > 0$. Hence a_n is strictly increasing.

Definition 1.9. We say that a sequence $\{a_n\}$ is **bounded** if there are $m, M \in \mathbb{R}$ such that

$$m \leq a_n \leq M$$

for all n. If we only have the inequality for M (resp. m) then $\{a_n\}$ is said to be **bounded from above** (resp. **bounded from below**).

It turns out that if we have an *increasing* sequence that is bounded from *above*, then it converges. Similarly for decreasing sequences bounded from below. This is the content of the next theorem.

Theorem 1.10 (Monotone Convergence Theorem)**.** *Suppose $\{a_n\}$ is both bounded and monotone. Then $\{a_n\}$ converges.*

Example. Define a sequence recursively by $a_1 = 1$, $a_{n+1} = 3 - \frac{1}{a_n}$. We want to prove that this sequence converges and to find its limit. To do this we prove that the sequence is increasing and bounded from above. Use induction on the statement '$0 < a_n < 3$ *and* $a_n < a_{n+1}$'.

The case for $n = 1$ is clear. Now suppose the statement is true for $n = k$ so that we have $0 < a_k < 3$ and $a_k < a_{k+1}$. Then it immediately follows that $a_{k+1} > 0$. Also,

$$\begin{aligned} 3 &> a_k \\ \frac{1}{a_k} &> \frac{1}{3} && \text{(since } a_k > 0) \\ \frac{-1}{a_k} &< \frac{-1}{3} \\ 3 - \frac{1}{a_k} &< 3 - \frac{1}{3} < 3 \\ a_{k+1} &< 3. \end{aligned}$$

Finally, we do the same trick for $a_k < a_{k+1}$:

$$\begin{aligned} a_k &< a_{k+1} \\ 3 - \frac{1}{a_k} &< 3 - \frac{1}{a_{k+1}} \\ a_{k+1} &< a_{k+2} \end{aligned}$$

Thus the proposition is true for all n and so the sequence $\{a_n\}$ converges by the Monotone Convergence Theorem.

Denote the limit of the sequence by $\ell > 0$ (why can it not be 0?). Notice that both $a_n \to \ell$ and $a_{n+1} \to \ell$ (why?). Hence by taking the limit in

$$a_{n+1} = 3 - \frac{1}{a_n}$$

we get

$$\ell = 3 - \frac{1}{\ell},$$

which rearranges to

$$\ell^2 - 3\ell + 1 = 0.$$

This has the solutions

$$\frac{3 \pm \sqrt{5}}{2},$$

but ℓ has to be unique so only one of them can be the correct limit. Notice that $(3 - \sqrt{5})/2 < 1$ whereas $a_n > 1$ for all $n > 1$. Hence we conclude that

$$\ell = \frac{3 + \sqrt{5}}{2}.$$

It is important to realise that the computation for the limit is only valid once we have established (e.g. by the MCT) that the sequence converges. Merely doing this computation for the limit is *not* enough to prove that the sequence converges.

Example. Let $a_n = \sqrt{n^2 + n} - n$. Notice that both $\sqrt{n^2 + n}$ and n diverge as $n \to \infty$, so Algebra of Limits does not apply. However, this sequence is in fact convergent. To see this we use a common trick of multiplying an expression with radicals by its conjugate (not to be confused with complex conjugate). That is, suppose we are faced with an expression of the form $\sqrt{a} - \sqrt{b}$, we then do

$$\sqrt{a} - \sqrt{b} = (\sqrt{a} - \sqrt{b})\frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}} = \frac{a - b}{\sqrt{a} + \sqrt{b}},$$

that is

$$\boxed{\sqrt{a} - \sqrt{b} = \frac{a - b}{\sqrt{a} + \sqrt{b}}.}$$

This expression is often easier to deal with. To go back to our example, we get

$$\begin{aligned}\sqrt{n^2+n}-n &= \sqrt{n^2+n}-\sqrt{n^2}\\ &= \frac{n^2+n-n^2}{\sqrt{n^2+n}+\sqrt{n^2}} && (\text{set } a=n^2+n \text{ and } b=n^2)\\ &= \frac{n}{n\sqrt{1+\frac{1}{n}}+n}\\ &= \frac{1}{\sqrt{1+\frac{1}{n}}+1} \to \frac{1}{2},\end{aligned}$$

since $\sqrt{x}$ is continuous at 1.

1.2 Series

Recommended problems from §11.2: **28, 36, 45, 48, 75, 88**.

The general idea for series is that we add the terms of a sequence together. It is however misleading to think of a series as simply an infinite sum since they do not generally behave as finite sums do (for example, we can only rearrange the terms of a series under certain conditions. We'll return to this point once we talk about absolute convergence). Instead, one should understand series as arising from a limiting process of adding more and more terms of a sequence together. To illustrate the construction of a sequence consider the following example of Maya the Bee and the bulldozers.

Example. Maya the Bee is stuck between two bulldozers and is about to be squashed! The bulldozers (initially at 0 and 1, respectively) move towards the middle both at a speed of 1 km/h. Maya frantically flies between the two bulldozers at a speed of 2 km/h and immediately bounces back towards the opposite direction upon touching one of the bulldozers. Maya is also initially at 0.

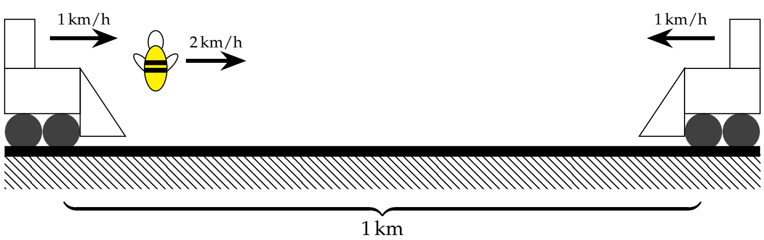


Figure 1.2: Maya the Bee being squashed by bulldozers.

The question we ask is: *what is the total distance travelled by Maya before she is squashed*?

Let x_n denote the distance travelled after n steps (by one step we mean a trip starting from one bulldozer until she hits the other one). With a little thought we can see that

$$x_1 = \frac{2}{3}.$$

This is because the total distance travelled by the second bulldozer and Maya has to be 1 after the first collision and Maya is travelling at twice the speed of the bulldozer. Alternatively, you can find the time of the collision by solving $2t = 1 - t$, which gives $t = 1/3$ so that the distance travelled by Maya is $2 \cdot 1/3$. For the next step via the same logic we get

$$x_2 = \frac{2}{3} + \frac{2}{9}.$$

Or you can again solve $\frac{2}{3} - 2t = \frac{1}{3} + t$ to arrive at the same result. In general we see

$$\begin{aligned} x_n &= \frac{2}{3} + \frac{2}{9} + \ldots + \frac{2}{3^n} \\ &= 2\left(\frac{1}{3} + \frac{1}{9} + \ldots + \frac{1}{3^n}\right) \\ &= 2\frac{\frac{1}{3}(1 - \frac{1}{3^{n+1}})}{1 - \frac{1}{3}} \\ &= 1 - \frac{1}{3^{n+1}} \to 1 \quad \text{as } n \to \infty. \end{aligned}$$

That is, the total distance travelled by Maya is 1. The corresponding series consists of the infinitely many individual distances (that Maya travels) added together. We then say that the sum of this series is 1.

It is possible to immediately see that 1 must be the final answer without doing any long computations. Can you figure out how?

In general we denote **series** by

$$a_1 + a_2 + a_3 + \ldots = \sum_{n=1}^{\infty} a_n.$$

Let us now formalise the idea of finding the sum of a series by computing the limit of the sequence of adding more and more of the individual terms together.

Definition 1.11. The N**-th partial sum** of $\sum_{n=1}^{\infty} a_n$ is

$$s_N = \sum_{n=1}^{N} a_n = a_1 + \ldots + a_N.$$

Moreover, if the sequence of partial sums $\{s_n\}$ converges then we say that the series converges and

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n.$$

Example. The series we encountered in the example with Maya is a specific case of a more general type of series called the *geometric series*. These are of the form

$$G = \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \dots$$

where $a \neq 0$. Let's examine how this series behaves for different r. First, if $r = 1$ then we're just summing up a constant sequence, i.e. $G = a + a + a + \dots$, which diverges. Otherwise we can look at the n-th partial sum of G:

$$s_n = a + ar + ar^2 + \dots + ar^n,$$

and multiply this by r,

$$rs_n = \quad ar + ar^2 + \dots + ar^n + ar^{n+1}.$$

Subtracting the second equation from the first yields

$$s_n(1-r) = a(1-r^{n+1}),$$

so that

$$s_n = \frac{a(1-r^{n+1})}{1-r}.$$

By using the limit (1.5) we conclude that s_n diverges if $|r| \geq 1$ and converges to $a/(1-r)$ otherwise. Hence, the geometric series satisfies

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

as long as $|r| < 1$.

Example. Consider $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. For series of this type it is useful to consider the partial fraction decomposition of the summands. We have

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Thus looking at the partial sums we notice that most of the terms cancel:

$$\begin{aligned}
s_1 &= \left(1 - \frac{1}{2}\right) \\
s_2 &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) \\
&= 1 - \frac{1}{3} \\
s_3 &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) \\
&= 1 - \frac{1}{4} \\
&\vdots \\
s_n &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= 1 - \frac{1}{n+1} \to 1.
\end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

This is called a *telescoping series* with the defining feature that each partial sum has a constant number of terms.

In general it is very difficult to find the sum of a series. Hence our focus will be on determining whether a given series converges or not. For this we will develop various tests for convergence. The simplest observation is the following:

Theorem 1.12. *If $\sum_n a_n$ converges then $a_n \to 0$.*

Notice that this theorem follows immediately from the observation that $a_n = s_n - s_{n-1} \to 0$ since both s_n and s_{n-1} converge to the same limit. This leads to the following

Test for divergence: Check if $a_n \to 0$. If **not** then $\sum_n a_n$ diverges.

For example, consider the sequence $\sum_{n=1}^{\infty}(1+\frac{2}{n})$ then for the general term we have $(1+\frac{2}{n}) \to 1 \neq 0$ so the series diverges. Notice that the converse of this theorem is not true in general. That is we can find series $\sum_n a_n$, for which $a_n \to 0$, that still diverge. One such series is the *harmonic series*:

$$H = \sum_{n=1}^{\infty} \frac{1}{n}.$$

Naturally $\frac{1}{n} \to 0$. We now show that the partial sums of H diverge. In each of the partial sums after every term of the form $1/2^i$ we group up the terms until the next power of two:

$$\begin{aligned}
s_2 &= 1 + \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) \\
s_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right).
\end{aligned}$$

For each of the terms in the brackets we can bound them from below by the last term:

$$\begin{aligned}
s_2 &= 1 + \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) \\
s_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\
&= 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right).
\end{aligned}$$

But then all the terms inside each set of brackets add up to half so

$$\begin{aligned} s_2 &> 1 + \frac{1}{2} \\ s_4 &> 1 + \frac{1}{2} + \frac{1}{2} \\ s_8 &> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}. \end{aligned}$$

We can apply the same logic to the 2^n-th term to get

$$s_{2^n} > 1 + \frac{n}{2}.$$

Taking the limit as $n \to \infty$ shows that the sequence of partial sums diverges, which means that the series $\sum_n \frac{1}{n}$ diverges.

We also note that series behave as expected under linear transformations.

Theorem 1.13. *Suppose $\sum a_n$ and $\sum b_n$ converge. Let $\alpha, \beta \in \mathbb{R}$. Then*

$$\sum_n (\alpha a_n + \beta b_n) = \alpha \sum_n a_n + \beta \sum_n b_n.$$

Next we consider series with positive terms. Then the sequence of partial sums is increasing (since $s_{n+1} = s_n + a_{n+1}$ so $s_{n+1} \geq s_n$ as $a_{n+1} \geq 0$). It follows from the Monotone Convergence Theorem that if $\{s_n\}$ is in addition bounded from above then it converges (and thus $\sum_n a_n$ also converges).

Example. Consider $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$. The terms in the series are positive so it remains to bound the partial sums from above. To do this we bound the general term from above by something for which the corresponding series converges. We have

$$\frac{1}{n^2} < \frac{2}{n^2 + n}. \qquad \text{(show this is true for } n > 1\text{)}$$

It follows that the partial sums of S are bounded (suppose $N > 1$):

$$\sum_{n=1}^{N} \frac{1}{n^2} < \sum_{n=1}^{N} \frac{2}{n^2 + n} \leq \sum_{n=1}^{\infty} \frac{2}{n(n+1)} = 2.$$

Therefore S converges (and is bounded from above by 2).

1.3 Integral Test

Recommended problems from §11.3: **14, 22, 27, 29, 30, 40, 44**.

Recall that integrals are defined as the limit of upper and lower Riemann sums (i.e. through sums that approach the value of the integral from above and below), see Figures 1.3 and 1.4. It is therefore possible, under certain conditions, to deduce the convergence of one from the other. More precisely this is known as the Integral Test.

Integral Test. *Suppose $f(x) \geq 0$ for $r \leq x < \infty$, where r is some integer. Also assume that $f(x)$ is decreasing. Then*

$$\sum_{k=r}^{\infty} f(k) \textit{ converges} \iff \int_r^{\infty} f(x)\,dx \textit{ converges.}$$

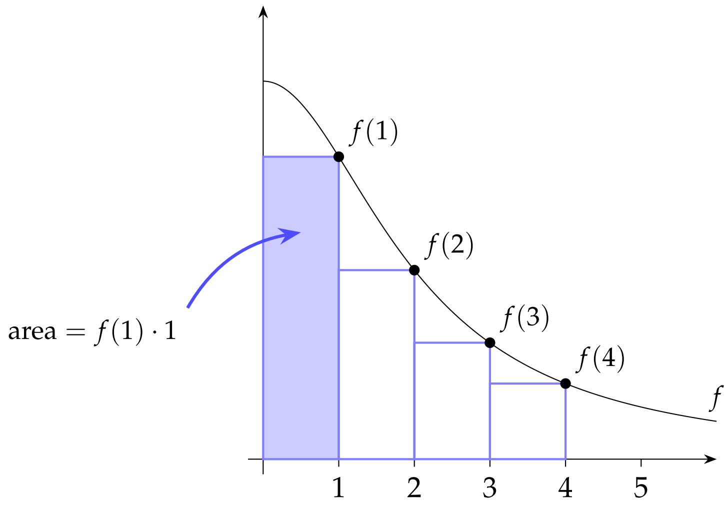


Figure 1.3: Lower sums

Let's give a sketch of the proof. From Figure 1.3 we see that

$$f(1) + f(2) + f(3) + f(4) \leq \int_0^4 f(x)\,dx.$$

In general we obtain an upper bound for the series as

$$\sum_{k=r+1}^{\infty} f(k) \leq \int_r^{\infty} f(x)\,dx. \tag{1.6}$$

Notice that we can use this bound to estimate series. Suppose we want to find an $N > 0$ such that $\sum_{n=1}^{N} f(n)$ (where $f(n) \geq 0$) is within an $\alpha > 0$ of the true value $\sum_{n=1}^{\infty} f(n)$. Then we are asking for which N is it true that

$$\sum_{n=1}^{\infty} f(n) - \sum_{n=1}^{N} f(n) < \alpha,$$

that is

$$\sum_{n=N+1}^{\infty} f(n) < \alpha.$$

By equation (1.6) it suffices to choose N so that

$$\int_N^{\infty} f(x)\,dx < \alpha,$$

and then

$$\sum_{n=1}^{\infty} f(n) - \sum_{n=1}^{N} f(n) \le \int_N^{\infty} f(x)\,dx < \alpha.$$

We can work similarly with upper Riemann sums to obtain a lower bound.

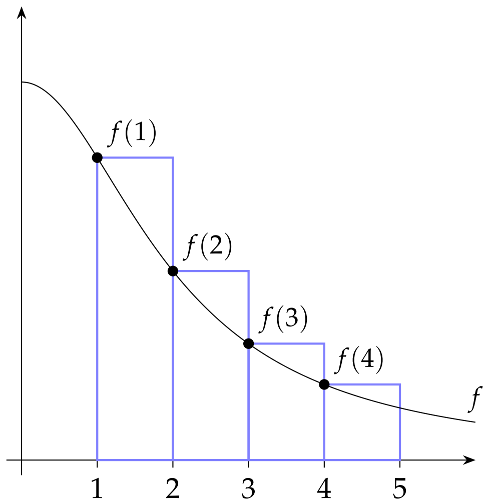


Figure 1.4: Upper sums

From Figure 1.4 we see that

$$\int_1^5 f(x)\,dx \le f(1) + f(2) + f(3) + f(4).$$

In general this becomes

$$\int_{r+1}^{\infty} f(x)\,dx < \sum_{n=r+1}^{\infty} f(n). \tag{1.7}$$

Working similarly as with equation (1.6), we can also estimate truncated series from below as

$$\sum_{n=1}^{\infty} f(n) - \sum_{n=1}^{N} f(n) \ge \int_{N+1}^{\infty} f(x)\,dx.$$

Example. Consider again the series $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$. The corresponding function is $f(x) = 1/x^2$ and $r = 1$. Notice that clearly $f(x)$ is positive and decreasing on $[1, \infty)$. It follows that the series S converges iff the integral of f converges:

$$\int_1^{\infty} \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^{\infty} = 1.$$

Thus the integral test tells us that S also converges. Moreover, from the observation (1.7) we know that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \geq 1.$$

The series in the above example is a specific case of a more general type of series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

Series of this type are called *p-series*. Let us now use the integral test to find the values of p for which the series converges. First of all if $p \leq 0$ then the sequence $n^{-p} \not\to 0$ and thus the p-series diverges. Otherwise $f(x) = x^{-p}$ is a positive decreasing function (since $p > 0$) on $[1, \infty)$ so we can apply the integral test. We have

$$\int_1^{\infty} x^{-p}\, dx = \begin{cases} [\ln x]_1^{\infty}, & \text{if } p = 1, \\ \frac{x^{1-p}}{1-p}, & \text{if } p \neq 1, p > 0. \end{cases}$$

In the case $p = 1$ we see immediately that the integral diverges since $\ln x \to \infty$ as $x \to \infty$. In the second case we note that $x^{1-p} \to 0$ as $x \to \infty$ if $1 - p < 0$ and diverges otherwise. Collecting all the cases together we conclude that

$$\boxed{\text{The } p\text{-series } \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges iff } p > 1.}$$

1.4 Comparison Test

Recommended problems from §11.4: **5, 6, 14, 26, 32, 43, 44, 46**.

Recall that for a series of positive terms the sequence of partial sums is increasing and thus convergent if it is bounded from above. For our next test for convergence we express this idea more simply in terms of the corresponding series.

Comparison Test. *Suppose $a_n, b_n \geq 0$ and that $a_n \leq b_n$ for all n. Then*

(i) $\sum_n b_n$ *converges* $\implies$ $\sum_n a_n$ *converges,*

(ii) $\sum_n a_n$ *diverges* $\implies$ $\sum_n b_n$ *diverges.*

We can now deduce much more quickly (compared to the end of last lecture) that $\sum_n n^{-2}$ converges by noting that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \sum_{n=1}^{\infty} \frac{2}{n(n+1)},$$

since $1/n^2 \leq 2/(n(n+1))$. Because the telescoping series converges to 2 it follows from the comparison test that the series on the LHS also converges. Of course we could also simply just note that it is a p-series with $p = 2$ and thus converges. The next example is a bit more interesting.

Example. Consider
$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}. \tag{1.8}$$
This is not a p-series, although it is not far off. One way to think about this series is that once n is big n^2 is huge compared to the 1 in the denominator and thus n^2 dominates. So in essence this series should behave roughly as the p-series with $p = 2$. We can then try to compare the general terms and notice that
$$\frac{1}{n^2+1} < \frac{1}{n^2}$$
for all n. Therefore by comparison test the series (1.8) converges.

Example. Now look at
$$\sum_{n=1}^{\infty} \frac{2}{n+1}. \tag{1.9}$$
Then since $2/(n+1) \geq 1/n$ and $\sum_n n^{-1}$ diverges (as it is the harmonic series), it follows by the comparison test that the series (1.9) diverges.

Consider now the series $S = \sum_n \frac{1}{2^n-1}$. Heuristically, this series should converge since when n is large 2^n is massive compared to 1 and as such it should dominate. Therefore the series resembles a geometric series with $r = 1/2$ which converges. We would like to compare S with this geometric series to prove convergence, but this is not possible since

$$\frac{1}{2^n-1} > \frac{1}{2^n}$$

for all positive n, so the comparison theorem does not apply. Yet, our intuition tells us that both of these series behave similarly. To use this logic in practice we introduce another test, which only takes into account the relative asymptotic behaviour of the general term of the series.

Limit Comparison Test. *Suppose $a_n, b_n > 0$ for all n, and assume $\frac{a_n}{b_n} \to \ell$ for some $\ell > 0$. Then*

$$\sum_n a_n \text{ converges} \iff \sum_n b_n \text{ converges.} \tag{1.10}$$

If, however, $\ell = 0$ (i.e. $\frac{a_n}{b_n} \to 0$) then we only have the weaker statement that

$$\sum_n b_n \text{ converges} \implies \sum_n a_n \text{ converges.}$$

Notice that the equivalence (1.10) is really saying that either both of the series converge or both diverge (i.e. if one converges then the other one has to converge as well, and if one of them diverges then the other one has to diverge, too). Also, in the case $\ell = 0$ we necessarily have that b_n is of higher order of magnitude than a_n so generally it is easier just to apply the comparison test instead.

Example. Let's now use the limit comparison test to prove $S = \sum_{n=1}^{\infty} 1/(2^n - 1)$ converges. We compare with $\sum_{n=1}^{\infty} 2^{-n}$. We find that

$$\frac{1}{2^n-1} \cdot 2^n = \frac{1}{1-2^{-n}} \to 1$$

as $n \to \infty$. Since $1 > 0$ and $\sum_{n=1}^{\infty} 2^{-n}$ converges (why?), it follows from the limit comparison test that S also converges.

Example. Let

$$S = \sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n} = \sum_n a_n.$$

This series is a bit deceitful. It might be tempting to try and compare it with $\sum_n n^{-1/2} = \sum_n b_n$ and to try and deduce that S diverges. This is not the case however! Moreover, you can check that $a_n/b_n \to 0$ so the limit comparison test does not apply (since $\sum_n b_n$ diverges). Instead we simplify the a_n's by using the trick with differences of radicals we saw before:

$$\begin{aligned} a_n &= \frac{\left(\sqrt{n+1}-\sqrt{n}\right)}{n} \cdot \frac{\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}} \\ &= \frac{n+1-n}{n\left(\sqrt{n+1}+\sqrt{n}\right)} \\ &= \frac{1}{n\left(\sqrt{n+1}+\sqrt{n}\right)}. \end{aligned}$$

Now, since $\sqrt{n+1} > 0$, we can simply ignore it in the denominator to obtain an upper bound:

$$a_n < \frac{1}{n\sqrt{n}} = n^{-3/2}$$

so by comparison test it follows that S converges (we compare it with p-series with $p = 3/2$).

Example. There's often more than one possibility for your choice of series to compare with. Take for example

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 - 1} = \sum_{n=1}^{\infty} a_n.$$

We again guess that we should compare this with $\sum_n n^{-3/2} = \sum_n b_n$ (why?). Instead of trying to come up with a delicate upper bound we simply apply the limit comparison test:

$$\frac{a_n}{b_n} = \frac{\sqrt{n}}{n^2 - 1} n^{3/2} = \frac{n^2}{n^2 - 1} \to 1 > 0.$$

Hence since $\sum_n b_n$ converges, by the LCT $\sum_n a_n$ converges.

Notice that we could also compare with $\sum_n n^{-5/4}$ and get

$$\frac{a_n}{n^{-5/4}} \to 0$$

and we can use the second part of LCT to conclude that $\sum_n a_n$ converges. On the other hand suppose we try to compare with $\sum_n n^{-7/4}$ (which also converges). Then, however,

$$\frac{a_n}{n^{-7/4}} \to \infty$$

so we cannot apply LCT.

1.5 Alternating Series

Recommended problems from §11.5: **15, 16, 20, 35**.

We'll now move on to consider more general series with both positive and negative terms. For such series there can be cancellation between the terms of the series and thus less decay is required from the individual terms. This is particularly easy to see in an important special case where the sign of consequent terms keeps alternating. We can write series of this type as

$$a_1 - a_2 + a_3 - a_4 + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} a_n,$$

where $a_n > 0$. We can then guarantee that a series of this form converges under fairly weak conditions on a_n.

Alternating Series Test. *Let* $A = \sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *be an alternating series, so that* $a_n > 0$ *for all* n*. Moreover, suppose the* a_n*'s satisfy*

(i) $a_{n+1} \leq a_n$ *for all* n *(so the sequence is decreasing),*

(ii) $\lim_{n\to\infty} a_n = 0.$

Then, A *converges.*

Notice that this test cannot be used to deduce divergence. That is, if you have an alternating series which does not satisfy all of the conditions of the test, then you cannot immediately decide that the series diverges. Instead you have to analyse it further. Next we give a sketch of the proof.

Proof. Let's look at the even partial sums of A:

$$s_{2N} = a_1 - a_2 + a_3 - a_4 + \ldots + a_{2N-1} - a_{2N}.$$

We can group the terms in various ways. First, notice that

$$s_{2N} = (a_1 - a_2) + (a_3 - a_4) + \ldots + (a_{2N-1} - a_{2N}).$$

Each of the quantities in brackets is positive so it follows that s_{2N} is an increasing sequence. On the other hand, we also have that

$$s_{2N} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \ldots - a_{2N} \leq a_1, \tag{1.11}$$

since again all the grouped quantities are positive. It follows that s_{2N} is both bounded from above and increasing and so it converges to A. But then, for the odd partial sums we note that

$$s_{2N+1} = s_{2N} + a_{2N+1}.$$

However from our assumptions we know that $a_{2N+1} \to 0$ and hence $s_{2N+1} \to A$. Since both subsequences, of even and odd partial sums, converge to the same limit it follows that $s_n \to A$, that is, the series A is convergent. □

From this proof we can see how to estimate alternating series. Working as with (1.11) we obtain

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n \leq a_1.$$

Or in general

$$\left|\sum_{n=N}(-1)^{n+1}a_n\right| \leq a_N. \tag{1.12}$$

Example. Recall that the harmonic series $\sum_n \frac{1}{n}$ diverges. Now look at $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \sum_{n=1}^{\infty}(-1)^{n+1}a_n$. First of all this is an alternating series since $a_n > 0$. Moreover a_n is clearly decreasing and satisfies $a_n \to 0$. Therefore we can apply the Alternating Series Test to conclude that $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ converges.

Example. What about

$$S = \sum_{n=1}^{\infty} \left(1 + \frac{2}{n}\right)(-1)^n = \sum_{n=1}^{\infty} (-1)^n a_n?$$

Well this is certainly an alternating series, but it does not satisfy the conditions of the AST. So what to do? It seems we forgot about the most basic test for divergence! Consider the even and odd subsequences of the general term:

$$\begin{aligned} a_{2n} &= \left(1 + \frac{2}{2n}\right)(+1) \to 1, \\ a_{2n+1} &= \left(1 + \frac{2}{2n+1}\right)(-1) \to -1. \end{aligned}$$

Since the two subsequences converge to a different limit it follows that the sequence a_n cannot converge. Therefore the series S diverges.

Example. Let's see how to estimate alternating series in practice. Let

$$S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{100^n}.$$

We wish to estimate S to 6 decimal places. To do this we want to use the fact that an alternating series starting from N is estimated in absolute value by a_N. Thus we need to find the first a_n such that $a_n < 10^{-6}$. We see

$$\begin{aligned} a_1 &= 0.01, \\ a_2 &= 0.0002, \\ a_3 &= 0.000003, \\ a_4 &= 0.00000004. \end{aligned}$$

Hence, $a_4 = 4 \cdot 10^{-8} < 10^{-6}$. It follows from the estimate (1.12) that

$$|a_1 - a_2 + a_3 - S| \le a_4 < 10^{-6},$$

because $|a_1 - a_2 + a_3 - S| = |\sum_{n=4}^{\infty} (-1)^{n+1} a_n|$. So we see that to within 6 decimal places S is approximated as

$$S \approx a_1 - a_2 + a_3 = 0.009803.$$

1.6 Absolute Convergence and Root and Ratio tests

Recommended problems from §11.6: **18, 20, 36, 37, 38**.

Recall that we showed that the series $\sum_n (-1)^{n+1} \frac{1}{n}$ converges whereas $\sum_n |(-1)^{n+1}| \frac{1}{n} = \sum_n \frac{1}{n}$ diverges. A series of this type is called conditionally convergent. This type of convergence is very weak. More precisely, we have

Definition 1.14. The series $\sum_n a_n$ is called **absolutely convergent** if $\sum_n |a_n|$ converges.

Definition 1.15. The series $\sum_n a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

Recall that it is important not to treat series as simply infinite sums. For example, we are only allowed to rearrange the terms of a series if it converges absolutely. Moreover, it turns out that for any conditionally convergent series its terms can be rearranged to sum to any finite number or even be made to diverge to either ∞ or $-\infty$.[4] In order to combine absolute convergence with other tests for convergence we need the following theorem.

Theorem 1.16.

$$\sum_n |a_n| \text{ converges } \implies \sum_n a_n \text{ converges.}$$

Example. Let's show that $S = \sum_n \frac{\sin n}{n^2}$ converges absolutely. We have

$$\frac{|\sin n|}{n^2} \leq \frac{1}{n^2}.$$

It follows from p-series and comparison test that $\sum_n \frac{|\sin n|}{n^2}$ converges, i.e. S converges absolutely. From Theorem 1.16 we see that S converges.

[4] If you are curious you can read more about this at https://en.wikipedia.org/wiki/Riemann_series_theorem.

We'll next introduce two tests that are easy to use given that you recognise the correct situation you are in.

Ratio Test. *Let* $S = \sum_{n=1}^{\infty} a_n$ *and suppose* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$. *If*

(i) $L < 1$ *then* S *converges absolutely,*

(ii) $L > 1$ *then* S *diverges,*

(iii) $L = 1$ *then the test is inconclusive.*

Example. Let

$$S = \sum_{n=1}^{\infty} \frac{2^n n!}{n^n}.$$

We want to apply ratio test to decide whether the series converges. We have

$$a_{n+1} = \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}, \quad a_n = \frac{2^n n!}{n^n}.$$

Hence

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n n!} \\
&= \frac{2(n+1)n^n}{(n+1)^{n+1}} \\
&= \frac{2n^n}{(n+1)^n} \\
&= 2\left(\frac{n+1}{n}\right)^{-n} \\
&= 2\left(1+\frac{1}{n}\right)^{-n} \\
&= 2e^{-n\ln(1+\frac{1}{n})}.
\end{aligned}$$

To compute this limit we apply L'Hôpital's Rule:

$$\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\ln(1+\frac{1}{n})}{1/n} = \lim_{n\to\infty}\frac{\left(\frac{-1}{n^2}\right)\frac{1}{1+\frac{1}{n}}}{\frac{-1}{n^2}} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1. \tag{1.13}$$

It follows that (since e^x is continuous)

$$\frac{a_{n+1}}{a_n} \to 2e^{-1} = \frac{2}{e} < 1,$$

since $e \approx 2.71828$. Since the limit is less than 1 it follows that the series converges by the ratio test.

As we see ratio test is easy to apply in cases where there's a lot of simplification between the ratio of consequent terms of the series, for example, when it involves products of quantities involving n, such as factorials.

Example. Consider
$$\sum_{n=1}^{\infty} \frac{n!(2n)!7^n}{(3n)!} = \sum_{n=1}^{\infty} a_n.$$
Look at the ratio of consequent terms:
$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+1)!(2n+2)!7^{n+1}}{(3n+3)!} \cdot \frac{(3n)!}{n!(2n)!7^n} \\ &= \frac{7(n+1)(2n+2)(2n+1)}{(3n+3)(3n+2)(3n+1)},\end{aligned}$$
since $\frac{(2n+2)!}{(2n)!} = \frac{(2n+2)(2n+1)(2n)!}{(2n)!} = (2n+2)(2n+1)$. Then
$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{7(1+\frac{1}{n})(2+\frac{2}{n})(2+\frac{1}{n})}{(3+\frac{3}{n})(3+\frac{2}{n})(3+\frac{1}{n})} \\ &= \frac{7\cdot 1\cdot 2\cdot 2}{3\cdot 3\cdot 3} \\ &= \frac{28}{27} > 1.\end{aligned}$$
Since the limit is greater than 1 it follows that the original series diverges.

Our final test looks at the n-th root of the general term of the series.

Root Test. *Let $S = \sum_n a_n$ and let $\lim_{n\to\infty} |a_n|^{1/n} = L$. If*

(i) $L < 1$ *then S converges absolutely,*

(ii) $L > 1$ *then S diverges,*

(iii) $L = 1$ *then the test is inconclusive.*

Example. Let
$$S = \sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} = \sum_{n=1}^{\infty} a_n.$$
Then, by applying the root test and using (1.13) we get
$$\sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \left(\frac{n}{n+1}\right)^n \to e^{-1} < 1.$$
Hence S converges. What if we change the exponent to n^3? We can start our investigation the same way. First denote this series by
$$\hat{S} = \sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^3}.$$
We get
$$\begin{aligned}\sqrt[n]{\left(\frac{n}{n+1}\right)^{n^3}} &= \left(\frac{n}{n+1}\right)^{n^2} \\ &= \exp\left(-n^2 \ln\left(1+\frac{1}{n}\right)\right).\end{aligned}$$

Ideally, we'd like to apply L'Hôpital's Rule again. Let's look at the limit of the derivates:

$$\lim_{x\to\infty} \frac{\frac{-1}{x^2}\cdot\frac{1}{1+\frac{1}{x}}}{\frac{-2}{x^3}} = \lim_{x\to\infty} \frac{x}{-2(1+\frac{1}{x})} = \infty.$$

Since the limit of the derivatives diverges we can't apply L'Hôpital's Rule! So what to do? Well one possibility is to first reason heuristically. Notice that the quantity we are exponentiating (that is, $\frac{n}{n+1}$) is less than 1. Also recall that $r^n \to 0$ for $|r| < 1$. In fact, for $0 < r < 1$ this is a decreasing sequence. So if we keep taking higher powers of $\frac{n}{n+1}$ we get smaller and smaller quantities. Since $n^3 > n^2$ we could try to compare this series with the one we already proved converges. To apply the comparison test we need to show that

$$\left(\frac{n}{n+1}\right)^{n^2} \geq \left(\frac{n}{n+1}\right)^{n^3},$$

we simplify to get

$$\exp\left(-n^2\ln(1+\frac{1}{n})\right) \geq \exp\left(-n^3\ln(1+\frac{1}{n})\right),$$

but this must hold since exp is an increasing function and $-n^2 \geq -n^3$ for $n \geq 1$ and $\ln(1+\frac{1}{n})$ is positive. It follows by comparison test that $\hat{S}$ converges. It is possible to do this by limit comparison test as well, but that is more tricky.

Remark. I wanted to make the comment I made during the lecture about the weakness of the ratio and root tests more precise. Consider first the p-series

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Of course we already know that this converges, but let's try to apply the ratio test and see what happens. We get

$$\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \frac{1}{(1+\frac{1}{n})^2} \to 1.$$

So the ratio test is inconclusive. The root test is slightly more powerful than the ratio test so let's apply it next.

$$\sqrt[n]{n^{-2}} = \left(\frac{1}{\sqrt[n]{n}}\right)^2 \to 1,$$

since $\sqrt[n]{n} \to 1$. Therefore the ratio test is also inconclusive!

On the other hand looking at e.g. the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

we can easily apply the ratio test

$$\frac{2^n}{2^{n+1}} = \frac{1}{2} < 1$$

so the series converges. Or let's apply the root test

$$\sqrt[n]{2^{-n}} = \frac{1}{2} < 1.$$

So again we see that the series converges.

In summary, these examples demonstrate that we need very rapid decay from the general term of the series before we get something useful out of the root and ratio tests.

We'll look at one more example which demonstrates this point.

Example. Let

$$S = \sum_{n=1}^{\infty} \frac{1}{e^{\sqrt{n}}} = \sum_{n=1}^{\infty} a_n.$$

The a_n's are exponentials in n so this should be a prime example for the application of the root test. Let's see what happens. We have

$$\sqrt[n]{|a_n|} = \frac{1}{e^{\sqrt{n}/n}} = \frac{1}{e^{1/\sqrt{n}}} \to \frac{1}{e^0} = 1.$$

Hence the root test is in fact inconclusive! It is also possible to show that the ratio test is inconclusive. Can you figure out what to do next? Try to prove the convergence or the divergence of S yourself before looking at the solution.

There are various ways to do this. First of all, this looks like something we could integrate so let's try to apply the integral test. For this we set $f(x) = e^{-\sqrt{x}}$. Then

$$\begin{aligned}
\int_1^{\infty} f(x)\,dx &= \int_1^{\infty} e^{-\sqrt{x}}\,dx \\
&= \int_1^{\infty} e^{-u}(2u)\,du && \text{(by subsitution } x = u^2\text{)} \\
&= -2\left[e^{-u}(u+1)\right]_1^{\infty} && \text{(by integration by parts)} \\
&< \infty
\end{aligned}$$

Therefore by the integral test S also converges. Another way to do this would be the comparison test. This is harder though since it might not be obvious what to compare with. To figure it out we could follow for example this train of thought: in S we have an exponential (even if it is for a lower power of n) so it should eventually overpower any polynomial decay. So with this in mind let's try to compare with some p-series, say, $p = 2$. For the limit comparison test we compute:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{1}{e^{\sqrt{n}}}}{\frac{1}{n^2}} &= \lim_{n\to\infty} \frac{n^2}{e^{\sqrt{n}}} \\
&= \lim_{n\to\infty} \frac{u^4}{e^u} && \text{(by the substitution } n = u^2\text{)} \\
&= \lim_{n\to\infty} \frac{4u^3}{e^u} && \text{(by L'Hôpital's Rule)} \\
&= \lim_{n\to\infty} \frac{4\cdot 3\cdot 2\cdot 1}{e^u} \to 0. && \text{(by repeated application of L'Hôpital's Rule)}
\end{aligned}$$

The limit is 0 but we recall that for limits of the type $a_n/b_n \to 0$ if $\sum_n b_n$ converges we can still deduce that $\sum_n a_n$ converges. It follows that S converges by the limit comparison test. Just for the fun of it, let's show one more way to prove that this series converges. For this we will need to borrow something we won't be covering until the next week of lectures. You can return to this example after learning about power series. In any case, we know that the power series for e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots,$$

which converges for any $x \in \mathbb{R}$. Notice that for positive x all the terms in the series are positive. So in particular if we remove all terms but one we obtain a lower bound for the exponential. Let's keep the $n = 4$ term:

$$e^x \geq \frac{x^4}{4!},$$

for $x > 0$. Rearrange this to

$$\frac{4!}{x^4} \geq e^{-x},$$

and now substitute $x = \sqrt{n}$ to get

$$\frac{4!}{n^2} \geq e^{-\sqrt{n}}.$$

Thus it follows from comparison test that

$$\sum_{n=1}^{\infty} e^{-\sqrt{n}} \leq \sum_{n=1}^{\infty} \frac{4!}{n^2},$$

which converges since it is p-series with $p = 2$. It follows from the comparison test that S converges.

As a final remark, this example again demonstrates the point that ratio and root test fail very easily. Even though we actually have an exponential in this example, it is still not enough for either of those tests to work!

1.7 More Examples for Testing Convergence of Series

Recommended problems from §11.7: **25, 33, 36, 37**.

We'll finish off our discussion of convergence tests by a series (pun intended) of further examples.

Example.

$$\sum_{n=1}^{\infty} \frac{5n + 2^n}{3^n} = \sum_{n=1}^{\infty} a_n.$$

Here we should be thinking that $5n$ is very small compared to 2^n once n is large so that the series should behave similarly to the geometric series $\sum_n (\frac{2}{3})^n$ and thus converge. We apply ratio test:

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{5(n+1)+2^{n+1}}{3^{n+1}} \cdot \frac{3^n}{5n+2^n} \\ &= \frac{1}{3} \cdot \frac{\frac{5(n+1)}{2^n}+2}{\frac{5n}{2^n}+1}.\end{aligned}$$

We now need to compute the limit of $n/2^n$. We use L'Hôpital's Rule:

$$\lim_{n\to\infty} \frac{n}{2^n} = \lim_{n\to\infty} \frac{1}{2^n \ln 2} = 0.$$

It follows that

$$\left|\frac{a_{n+1}}{a_n}\right| \to \frac{1}{3} \cdot \frac{0+2}{0+1} = \frac{2}{3} < 1.$$

Hence by the ratio test the series converges. Can you prove that this series converges by using the comparison test?

In this case it would also be possible to separate the series in to two to get

$$\sum_{n=1}^{\infty} \frac{5n+2^n}{3^n} = \sum_{n=1}^{\infty} \frac{5n}{3^n} + \sum_{n=1}^{\infty} \frac{2^n}{3^n},$$

and then deal with each of them separately. This is allowed since both of the resulting series converge. In general though one has to be very careful about splitting series into parts as it's very easy to end up with divergent series.

Example. Look at the following argument:

$$\begin{aligned}0 &= 0+0+0+\dots \\ &= (1-1)+(1-1)+(1-1)+\dots \\ &= 1-1+1-1+1-1+\dots \\ &= 1+(-1+1)+(-1+1)+(-1+1)+\dots \\ &= 1.\end{aligned}$$

Can you see where the mistake is? Let's examine the series

$$1-1+1-1+1-1+\dots = \sum_{n=1}^{\infty} (-1)^{n+1}.$$

The $2n$-th partial sum of this series is

$$s_{2n} = (1-1)+(1-1)+\dots+(1-1) = 0,$$

and similarly

$$s_{2n+1} = (1-1)+(1-1)+\dots+(1-1)+1 = 1.$$

Thus the sequence of partial sums diverges and so the corresponding series diverges. Hence our whole argument that $0 = 1$ is rubbish (as it should be)!

Example. Let
$$S=\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 5\cdot 8\cdots(3n-1)}=\sum_{n=1}^{\infty}a_n.$$
The quantities in the numerator and denominator resemble factorials. Thus we try to apply the ratio test:
$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right|&=\frac{1\cdot 3\cdots(2n+1)}{2\cdot 5\cdots(3n+2)}\cdot\frac{2\cdot 5\cdots(3n-1)}{1\cdot 3\cdots(2n-1)}\\&=\frac{2n+1}{3n+2}\\&=\frac{2+\frac{1}{n}}{3+\frac{2}{n}}=\frac{2}{3}<1.\end{aligned}$$
So the series converges.

Example. Look at
$$\sum_{n=1}^{\infty}\frac{1}{n^{1+\frac{1}{n}}}.$$
Here the exponent approaches 1 so we could think that eventually this series will look almost like the harmonic series. Hence we expect that it diverges. Let's prove this by comparing with the harmonic series. We have
$$\frac{1}{n^{1+1/n}}\cdot n=\frac{1}{n^{1/n}}\to 1.$$
Thus by limit comparison test the original series diverges.

Example. Suppose
$$S=\sum_{n=2}^{\infty}\frac{1}{(\ln n)^{\ln n}}=\sum_{n=2}^{\infty}a_n.$$
Now by applying $a^b=e^{b\ln a}$ twice we get
$$(\ln n)^{\ln n}=e^{\ln n\ln(\ln n)}=n^{\ln\ln n}.$$
Thus
$$S=\sum_{n=2}^{\infty}\frac{1}{n^{\ln\ln n}}.$$
This looks quite difficult to estimate, but it is actually possible to do it by applying the integral test (try it!). However, by making a simple observation we can determine the convergence easily. Since $\ln\ln n$ is an increasing function then eventually it is bigger than any constant. For example, there is an n_0 such that for $n>n_0$ we have $\ln\ln n\geq 2$, that is
$$n^{\ln\ln n}\geq n^2$$
so
$$\frac{1}{n^2}\geq\frac{1}{n^{\ln\ln n}}.$$

It follows from the comparison test that

$$\sum_{n=n_0}^{\infty} \frac{1}{n^2} \geq \sum_{n=n_0}^{\infty} \frac{1}{n^{\ln \ln n}}. \tag{1.14}$$

Since only the tail of the series matters for convergence, we conclude that also S converges. To see this more precisely add $\sum_{n=2}^{n_0-1} a_n$ (a finite quantity) to both sides of equation (1.14) to obtain an upper bound for S.

Example. Consider now

$$\sum_{n=1}^{\infty} n \sin(\frac{1}{n}).$$

This looks a bit tricky to estimate since $\sin \frac{1}{n}$ decays as $n \to \infty$. Now it is good to recall the useful approximation that $\sin x \approx x$ when x is small. Thus for large n it seems that n and $\sin \frac{1}{n}$ cancel out and the terms become roughly 1. This would hint at that the series diverges. This is easily checked by computing the limit:

$$\lim_{n\to\infty} n \sin \frac{1}{n} = \lim_{n\to\infty} \frac{\sin \frac{1}{n}}{1/n} = \lim_{n\to\infty} \frac{\frac{-1}{n^2} \cos \frac{1}{n}}{-1/n^2} = \cos 0 = 1,$$

by L'Hôpital's Rule (it is of type $0/0$). Since this limit is nonzero, the series diverges.

Example. Now we look at a more complicated example. Define

$$S = \sum_{n=1}^{\infty} (\sqrt[n]{2} - 1).$$

First of all, since $\sqrt[n]{2} - 1 \to 0$ we have no easy way out. You should verify that the ratio test and the root test are not useful in this case. One way to see what happens is to compare with the harmonic series:

$$\lim_{n\to\infty} \frac{2^{\frac{1}{n}} - 1}{1/n} = \lim_{n\to\infty} \frac{(\frac{-1}{n^2}) 2^{\frac{1}{n}} \ln 2}{-1/n^2} = \lim_{n\to\infty} 2^{\frac{1}{n}} \ln 2 \to \ln 2 \neq 0,$$

and thus S diverges by the limit comparison test. Now how did we come up with this comparison? Well in general it is one of the easiest and most useful comparisons to try for any series that you suspect is divergent. However, by borrowing again from next week a little, we arrive at the following argument. By the series for e^x we have

$$2^{\frac{1}{n}} = e^{\frac{1}{n} \ln 2} = 1 + \frac{\ln 2}{n} + \frac{\ln^2 2}{2!\, n^2} + \ldots$$

By positivity we obtain

$$2^{\frac{1}{n}} \geq 1 + \frac{\ln 2}{n}.$$

We can then use this bound to estimate S from below (this is abusing the notation a bit since the series are divergent, but you should understand the following statements to really apply to the partial sums) as

$$\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1) \geq \sum_{n=1}^{\infty} \frac{\ln 2}{n}.$$

So we see that S is bounded from below by the harmonic series times $\ln 2$. Therefore we can conclude by the comparison test that S diverges.

Finally, note that it is *not* allowed to try and separate this sum into two pieces and to deduce that the series diverges since each of the components does. Make sure you understand why this argument fails.

Finally, we finish off with a simple example where we take the previous series, but raise the terms to the n-th power.

Example. Let

$$\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)^n = \sum_{n=1}^{\infty} a_n.$$

Since the quantity inside the radical tends to 0, we can think of it as something very small. Taking a high power of a small number makes it much smaller. Therefore we might expect that this series in fact converges. To see this apply the root test:

$$\sqrt[n]{|a_n|} = \sqrt[n]{2} - 1 \to 0 < 1.$$

Hence by the root test this series converges.

1.8 Power Series

Recommended problems from §11.8: **20, 24, 37, 41, 42**.

We now start putting our knowledge of convergence of series into practice. We'll learn that it is possible to consider certain type of series as functions and, in particular, that these functions have very nice properties (infinitely differentiable etc.). We'll also learn how to express many familiar functions (for example sin, cos, exp) as such series.

In general power series are series of the type

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots,$$

for some fixed $c \in \mathbb{R}$. This is called a *power series centred at* c. In class we worked with $c = 0$, but here in the notes I will write the same definitions for the general case.

Definition 1.17. The **radius of convergence** of a power series $S = \sum_{n=0}^{\infty} a_n(x-c)^n$ is a number $R \geq 0$ (possibly ∞) such that (when $0 < R < \infty$)

(i) if $|x-c| < R$ then S converges (absolutely),

(ii) if $|x-c| > R$ then S diverges.

Moreover, if $R = \infty$ then S converges for all $x \in \mathbb{R}$. Also, all power series converge for $x = c$, if this is the only value of x for which S converges then we say $R = 0$.

The total set of points for which S converges is called the **interval of convergence**, and it necessarily contains $(c-R, c+R)$.

In other words a power series converges absolutely if $x \in (c-R, c+R)$. However, the above definition does not say anything about the behaviour of the power series at the end points of the interval of convergence, $x = c \pm R$. So for $x = c+R$ or $x = c-R$ the series could diverge or converge at both end points or any combination of these. In exercises, one has to check these separately by using some other tests of convergence.

An effective way to find the radius of convergence of a power series is ratio test. Let's work out some examples.

Example. Let

$$S = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} a_n x^n.$$

By ratio test

$$\left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{1}{n+1}|x| \to 0 < 1,$$

for any $x \in \mathbb{R}$. Therefore by ratio test S converges for all $x \in \mathbb{R}$ and therefore $R = \infty$.

Example. Now consider

$$\sum_{n=0}^{\infty} n!\, x^n.$$

Then
$$\left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right| = (n+1)|x| \to \infty$$
for any $0 \neq x \in \mathbb{R}$. It follows that the series converges only for $x = 0$.

Example. Next suppose that we modify the previous series to get
$$S = \sum_{n=0}^{\infty} n!\,x^{n^2}.$$
Then
$$\left|\frac{(n+1)!\,x^{(n+1)^2}}{n!\,x^{n^2}}\right| = (n+1)|x|^{n^2+2n+1-n^2} = (n+1)|x|^{2n+1}. \tag{1.15}$$
Recall the limit $\lim_{n\to\infty} r^n = 0$ if $0 \leq r < 1$ and which diverges for $r > 1$. Use this to prove that the limit of (1.15) is 0 for $|x| < 1$ and ∞ for $|x| > 1$. It follows that the radius of convergence of S is 1.

Finally we look at a familiar example in a new context.

Example. Suppose
$$S = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$
As an exercise, prove that the radius of convergence of S is 1. Let's see what happens at the end points of the interval of convergence. For $x = 1$ we obtain the harmonic series
$$\sum_n \frac{1}{n},$$
which of course diverges. On the other hand for $x = -1$, we have an alternating series
$$\sum_n \frac{(-1)^n}{n},$$
which we've seen converges by the alternating series test.

1.9 Power Series as Functions

Recommended problems from §11.9: **23, 28, 41**.

Now we get to the main idea behind introducing power series: *within the radius of convergence, we can treat a power series as a function of x*. This allows us to express complicated functions in a simpler form. In turn, expressing functions as a power series lets us prove powerful theorems about all such functions at once. Not all functions, however, have a power series expansion. It is still possible, for some of them, to approximate the function by a truncated power series. We'll discuss this in the context of Taylor's Theorem later on. Moreover, even if a function has a power series it might only represent the function on a small interval (indeed, this happens when R is finite). So to fully express the function we need to consider multiple power series.

For now, our main goal will be to see tricks and methods we can apply to deduce power series for more and more functions. As our most fundamental power series we recall an old example of the geometric series.

Example. We've seen that the geometric series,

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x},$$

converges to its sum if $|x| < 1$ and diverges otherwise. It follows that this is a power series with radius of convergence 1. In other words, if we let f be the function

$$f(x) = \frac{1}{1-x},$$

then for $x \in (-1, 1)$, f can be represented by a power series

$$f(x) = \sum_{n=0}^{\infty} x^n.$$

It is straightforward to see also that

$$\sum_{n=0}^{\infty} (x-c)^n$$

converges for $|x - c| < 1$, that is $x \in (c-1, c+1)$.

We'll next see how to do basic manipulations of power series (in order to deduce power series for more functions). First though, just a simple remark on multiplying generic series together.

Remark. For two series

$$\sum_{n=0}^{\infty} a_n \text{ and } \sum_{n=0}^{\infty} b_n,$$

their product is a series

$$\sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} c_n,$$

where the coefficients c_n are given by

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} a_{n-k} b_k = a_0 b_n + a_1 b_{n-1} + \ldots + a_{n-1} b_1 + a_n b_0.$$

Theorem 1.18. [5] *Let $\sum_{n=0}^{\infty} a_n (x-c)^n$ and $\sum_{n=0}^{\infty} b_n (x-c)^n$ be power series with radii of convergence r, s, respectively. Then the following arithmetic operations on power series also result in power series:*

[5] I'll write down the full theorem here even though the last parts will be covered in the next lecture.

Linear Combinations: *Suppose $\alpha, \beta \in \mathbb{R}$ are constants, then the linear combination of power series is also a power series,*

$$\sum_{n=0}^{\infty} \alpha a_n (x-c)^n + \sum_{n=0}^{\infty} \beta b_n (x-c)^n = \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n)(x-c)^n.$$

This power series has a radius of convergence at least $\min(r, s)$.

Multiplication: *The product of two power series is a power series,*

$$\sum_{n=0}^{\infty} a_n (x-c)^n \sum_{n=0}^{\infty} b_n (x-c)^n = \sum_{n=0}^{\infty} c_n (x-c)^n,$$

where the coefficients c_n are defined as above by

$$c_n = \sum_{k=0}^{n} a_k b_{n-k},$$

and the radius of convergence of the resulting product is again at least $\min(r, s)$.

Example. Generally when working with concrete examples it is easier just to compute the product by hand by looking at all the possible ways to get each power of x in the product. Try checking the following yourself:

$$(1 + x + x^2 + x^3 + x^4 + \ldots)(1 + x^2 + x^4 + \ldots) = 1 + x + 2x^2 + 2x^3 + 3x^4 + \ldots.$$

Notice that we can only compute terms up to x^4 from the above, unless we write the series down further.

Division: *If $b_0 \neq 0$ then the quotient of the power series is also a power series,*

$$\frac{\sum_{n=0}^{\infty} a_n x^n}{\sum_{n=0}^{\infty} b_n x^n} = \sum_{n=0}^{\infty} d_n x^n.$$

We can compute the coefficients d_n recursively by comparing coefficients in:

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n \sum_{n=0}^{\infty} d_n x^n.$$

Let's see an example.

Example.

$$\frac{1 + x + x^2 + x^3 + x^4 + \ldots}{1 - x^2 + x^4 - \ldots} = \sum_{n=0}^{\infty} d_n x^n$$

$$1 + x + x^2 + x^3 + x^4 + \ldots = (d_0 + d_1 x + d_2 x^2 + d_3 x^3 + d_4 x^4 + \ldots)(1 - x^2 + x^4 - \ldots).$$

Compare coefficients to get

$$\begin{aligned}
x^0 &: & 1 &= d_0 \cdot 1 & &\implies d_0 = 1, \\
x^1 &: & 1 &= d_1 \cdot 1 & &\implies d_1 = 1, \\
x^2 &: & 1 &= d_2 - d_0 & &\implies d_2 = 2, \\
x^3 &: & 1 &= d_3 - d_1 & &\implies d_3 = 2.
\end{aligned}$$

So

$$\frac{1 + x + x^2 + x^3 + x^4 + \ldots}{1 - x^2 + x^4 - \ldots} = 1 + x + 2x^2 + 2x^3 + \ldots$$

Aside: Proof by Induction

Due to this causing confusion to many students I decided to write a brief explanation-about mathematical induction. **Notice that this material is *non-examinable*.** This is useful for your general mathematical knowledge. Most of the examples in this note are from Martin Liebeck, *A Concise Introduction to Pure Mathematics*.

We've already seen induction used in some examples in class, but let's make everything a bit more precise. In its essence, the method of induction is a way of proving statements about integers. The way it works is that we first prove that our statement, denote it by $P(n)$, holds for some small integer, e.g. $n = 1$. This is called the *base case*. That is we have to show that $P(1)$ is true. This is usually trivial. After that comes the actual meat of induction: the *inductive step*. In the inductive step we assume that the statement is true for some integer k (that is we assume $P(k)$ holds). Then we have to deduce the validity of $P(k+1)$ from this. It then follows that our statement is true for all integers $n \geq 1$ (or whatever we choose as our base case). You can think of this as dominoes falling. The dominoes represent integers and knocking them over represents that the statement is true for that particular integer. In induction we then do the following: we prove that the first domino gets knocked over, and we also prove that if any given domino gets knocked over then necessarily it also knocks over the next one. Thus as all the dominoes fall, so also our statement becomes true for all integers.

Method of Induction. *Let $P(n)$ be a statement for any positive integer n. If we prove the following:*

(i) *$P(1)$ is true;*

(ii) *for all k, if $P(k)$ is true then $P(k+1)$ is also true;*

then $P(n)$ is true for all $n \geq 1$.

Remark. Without loss of generality we can obtain the same theorem by starting from any integer ≥ 0. There is also something called the *strong induction* where instead of assuming the statement is true for just $P(k)$, we assume it for all integers up to k (or some subset of these). We will not get into these more general forms in this note.

Let's now do some examples.

Example. Let $P(n)$ be the statement "$n < 2^n$" for all $n \geq 1$. We prove this by induction.

The base case is $P(1)$, that is $1 < 2^1$, which is clearly true. Now for the inductive step we assume that $P(k)$ is true, so we have $k < 2^k$. We then have to deduce $P(k+1)$ from this. We start by adding 1 to both sides of the inequality

$$k + 1 < 2^k + 1,$$

but then we know that $1 < 2^k$ for any $k \geq 1$ (since 2^x is increasing). Thus in the inequality we can bound the RHS from above by $2^k + 2^k$, i.e.

$$k + 1 < 2^k + 1 < 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}.$$

Hence we have obtained $P(k+1)$. By induction it follows that $n < 2^n$ for all $n \geq 1$.

Example. Let's try a different kind of example. Suppose $P(n)$ is the statement *"the sum of the first n odd integers is n^2"*.

Again, the base case is easy. $P(1)$ is $1 = 1^2$, which is of course true. In order to proceed further we have to be able to write down the sum for $P(k)$. To do this we need to understand how to write the sum of first k odd integers. First question to ask is: what is the k-th odd integer? After a little bit of thinking it should be clear that this is $2k-1$ (which simplifies to 1 for $k = 1$). Hence $P(k)$ becomes

$$1 + 3 + 5 + \ldots + (2k-1) = k^2.$$

In order to obtain $P(k+1)$ we want to have the sum of the first $k+1$ odd integers on the LHS, so we add $2(k+1) - 1 = 2k+1$ to both sides of the equation. This yields

$$1 + 3 + 5 + \ldots + (2k-1) + (2k+1) = k^2 + 2k + 1.$$

But then we notice that the RHS factorises as $(k+1)^2$. Hence $P(k+1)$ follows and thus by induction we deduce that $P(n)$ is true for all $n \geq 1$. That is, we have that the sum of the first n odd integers is n^2, i.e.

$$1 + 3 + 5 + \ldots + (2n-1) = n^2,$$

for any $n \geq 1$.

Here are some examples for you to try:

(i) prove that if $p > -1$ then $(1+p)^n \geq 1 + np$, deduce that $3^n > n$ from this;

(ii) prove the formula for the sum of n first squares:

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6};$$

(iii) let A be the matrix $\left(\begin{smallmatrix} 1 & 2 \\ 0 & -1 \end{smallmatrix}\right)$. Prove that for all $n \geq 1$,

$$A^n = \begin{pmatrix} 1 & 1 + (-1)^{n+1} \\ 0 & (-1)^n \end{pmatrix};$$

(iv) prove that the sequence $a_{n+1} = (a_n^2 + 1)/2$, $a_1 = 2$ is increasing. Does the sequence converge? If it does, find its limit, if not explain why.

Apart from the operations we saw last time, another useful technique is the composition of functions (i.e. composition of power series). Let's do two examples.

Example. Let

$$f(y) = \frac{1}{1-y} = 1 + y + y^2 + y^3 + y^4 + \ldots,$$
$$g(x) = -x^2.$$

Then

$$\begin{aligned}\frac{1}{1+x^2} &= \frac{1}{1-(-x^2)} \\ &= f(g(x)) \\ &= \sum_{n=0}^{\infty} (-x^2)^n \\ &= \sum_{n=0}^{\infty} (-1)^n x^{2n} \\ &= 1 - x^2 + x^4 - x^6 + \ldots\end{aligned}$$

What is the radius of convergence of this series? Let's use ratio test:

$$\left| \frac{(-1)^{n+1} x^{2n+2}}{(-1)^n x^{2n}} \right| = |x|^2 < 1.$$

It follows that the radius of convergence is 1.

Example. Let

$$f(y) = \frac{1}{1+y} = 1 - y + y^2 - y^3 + y^4 - \ldots,$$
$$g(x) = x + x^2.$$

Then

$$\begin{aligned}f(g(x)) &= 1 - (x+x^2) + (x+x^2)^2 - (x+x^2)^3 + (x+x^2)^4 - \ldots \\ &= 1 - x - x^2 + x^2 + 2x^3 + x^4 - x^3 - 3x^4 - 3x^5 - x^6 + x^4 + 4x^5 + 6x^6 + 4x^7 + x^8 + \ldots \\ &= 1 - x + x^3 - x^4 + \ldots\end{aligned}$$

Notice from the above expansions we can only compute the composition up to the x^4 term (unless we compute more terms initially).

On the other hand, we also have

$$1 - x^3 = (1-x)(1+x+x^2).$$

It then follows that

$$\begin{aligned}\frac{1}{1+x+x^2} &= \frac{1-x}{1-x^3} \\ &= (1-x)\sum_{n=0}^{\infty} \left(x^3\right)^n \\ &= (1-x)(1 + x^3 + x^6 + \ldots) \\ &= 1 - x + x^3 - x^4 + x^6 - x^7 + \ldots\end{aligned}$$

We also immediately know that the radius of convergence of the resulting series is at least 1.

Next we'll see that power series can be differentiated and integrated term by term within the radius of convergence.

Theorem 1.19. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Then for $x \in (-R, R)$ the function $f(x)$ is infinitely differentiable. Moreover, we have*

$$f'(x) = \frac{d}{dx} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{d}{dx} a_n x^n = \sum_{n=1}^{\infty} a_n n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

$$\int_0^x f(t)\, dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n\, dt = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \dots$$

Furthermore, both of these power series have a radius of convergence exactly R.

Here's an example.

Example. How to find a power series for $\frac{1}{(1-x)^2}$? Notice that

$$\frac{d}{dx} \frac{1}{1-x} = \frac{1}{(1-x)^2}.$$

Thus we can use geometric series and the previous theorem to differentiate term by term:

$$\begin{aligned} \frac{1}{(1-x)^2} &= \frac{d}{dx} \frac{1}{1-x} \\ &= \frac{d}{dx} \sum_{n=0}^{\infty} x^n \\ &= \sum_{n=0}^{\infty} \frac{d}{dx} x^n \\ &= \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + \dots \end{aligned}$$

By Theorem 1.19 the series we obtained also has a radius of convergence 1.

Example. We can also try to integrate. We know that

$$\int_0^x \frac{1}{1+t} dt = \ln(1+x) - \ln(1) = \ln(1+x).$$

Thus we can again use geometric series and integrate term by term to obtain

$$\begin{aligned} \ln(1+x) &= \int_0^x \frac{1}{1+t}\, dt \\ &= \int_0^x \sum_{n=0}^{\infty} (-1)^n t^n\, dt \\ &= \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n\, dt \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \end{aligned}$$

By reindexing we can write this as

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}.$$

Check that this gives exactly the same series as the equation above. Moreover, by Theorem 1.19 the radius of convergence is again 1.

Example. We want to find power series for $\arctan x$. To do this we recall the following elementary fact

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}.$$

Hence, as before,

$$\arctan x = \int_0^x \frac{1}{1+t^2}\, dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n}\, dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

What is the radius of convergence of this series?

Example. Find the interval of convergence (that is, all values of x for which the power series converges) of

$$S = \sum_{n=0}^{\infty} \frac{(3x-2)^n}{\sqrt{n}+2}.$$

First of all notice that this is not exactly in the form of a power series because of the coefficient on x. After factoring it out we get the power series representation

$$S = \sum_{n=0}^{\infty} \frac{3^n}{\sqrt{n}+2} \left(x - \frac{2}{3}\right)^n.$$

Hence S is a power series centered at $2/3$ with $a_n = 3^n/(\sqrt{n}+2)$. Then let's apply ratio test to find the radius of convergence:

$$\begin{aligned}\left| \frac{a_{n+1}}{a_n} \cdot \frac{(x-\frac{2}{3})^{n+1}}{(x-\frac{2}{3})^n} \right| &= \frac{3^{n+1}(\sqrt{n}+2)}{(\sqrt{n+1}+2)3^n} \left| x - \frac{2}{3} \right| \\ &= 3\left| x - \frac{2}{3} \right| \frac{\sqrt{n}+2}{\sqrt{n+1}+2} \to 3\left| x - \frac{2}{3} \right|.\end{aligned}$$

Thus in order for the series to converge we need that $3\left|x - \frac{2}{3}\right| < 1$, so the radius of convergence is $1/3$. More precisely, the power series converges for x which satisfy

$$\begin{gathered}\left| x - \frac{2}{3} \right| < \frac{1}{3} \\ \frac{-1}{3} < x - \frac{2}{3} < \frac{1}{3} \\ \frac{1}{3} < x < 1.\end{gathered}$$

To find the interval of convergence we also need to see what happens at the end points. For $x = 1$ we get

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n}+2},$$

which diverges (can you prove this? be careful if you use the comparison test (you are trying to deduce divergence)!). On the other hand for $x = 1/3$ we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}+2},$$

which converges by the alternating series test. It follows that the interval of convergence is $[\frac{1}{3}, 1)$. In other words the power series converges for x with $\frac{1}{3} \leq x < 1$.

Next we'll do an example with partial fractions.

Example. Let
$$f(x) = \frac{2x+3}{x^2+3x+2}.$$
Express f as a power series about 0 and find its interval of convergence.

First we have to factorise the denominator in order to use partial fractions. It's easy to see that
$$x^2+3x+2 = (x+1)(x+2)$$
(by noticing e.g. that -1 is a root). Then to decompose f with partial fractions we set
$$\begin{aligned} f(x) &= \frac{2x+3}{(x+1)(x+2)} \\ &= \frac{A}{x+1} + \frac{B}{x+2}, \end{aligned}$$
for some A and B. To find the coefficients we put everything under a common denominator and compare coefficients, i.e.
$$\frac{2x+3}{(x+1)(x+2)} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)},$$
so that we get the following system of linear equations:
$$\begin{cases} A+B=2 \\ 2A+B=3. \end{cases}$$
Subtracting the first equation from the second gives $A=1$, which we can substitute back to either equation to get $B=1$, as well. Thus
$$f(x) = \frac{1}{x+1} + \frac{1}{x+2}.$$
In order to use geometric series we simplify f a bit further:
$$\begin{aligned} f(x) &= \frac{1}{1-(-x)} + \frac{1}{2(1-(-\frac{x}{2}))} \\ &= \sum_{n=0}^{\infty}(-1)^n x^n + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{2^n}. \end{aligned}$$
Within the radius of convergence we can add the series together to get
$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty}(-1)^n \left(1+\frac{1}{2^{n+1}}\right) x^n \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{2^{n+1}+1}{2^{n+1}} x^n. \end{aligned}$$
Since the radius of convergence of both power series is 1 it follows that the radius of convergence of the resulting series is at least 1. Let's compute it. By the ratio test,
$$\left|\frac{(-1)^{n+1}}{(-1)^n}\cdot\frac{(2^{n+2}+1)2^{n+1}}{2^{n+2}(2^{n+1}+1)}\cdot\frac{x^{n+1}}{x^n}\right| = \frac{|x|}{2}\frac{2^{n+2}+1}{2^{n+1}+1} \to |x| < 1. \qquad \text{(why?)}$$

So the radius of convergence is exactly 1. To find the interval of convergence we need to check the end points.

For $x = 1$ we get

$$\sum_{n=0}^{\infty}(-1)^n\left(1+\frac{1}{2^{n+1}}\right) = \sum_{n=0}^{\infty} a_n.$$

While this is an alternating series, the alternating series test does not apply since the limit of the positive part is not zero. Let's look at the general term of the series: we notice that

$$\begin{aligned} a_{2n} &\to 1, \\ a_{2n+1} &\to -1. \end{aligned}$$

It follows that $a_n \not\to 0$, so the power series diverges for $x = 1$. On the other hand, for $x = -1$ we just have

$$\sum_{n=0}^{\infty}\left(1+\frac{1}{2^n}\right),$$

which of course diverges. Thus the interval of convergence is $(-1, 1)$.

Partial fractions with just linear factors are pretty straightforward, and this is what we most often encounter. It'll be good to know what to do with non-linear factors though. In general, the idea is that if the denominator has an n-th degree factor, we have to consider an $n-1$-th degree factor in the numerator. Otherwise the technique is the same (just compare coefficients). Use this to find the following partial fraction decomposition:

$$\frac{1}{(x^2+1)(x+3)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+3}.$$

Check your answer by just combining the terms under a common denominator.

Before we move on to the next section we put together a few things we've learnt so far in order to estimate integrals.

Example. Let

$$I = \int_0^{0.5} \frac{1}{1+x^3}dx.$$

Estimate the value of I to 3 decimals.

First we notice that the integrand has the following power series expansion

$$\frac{1}{1+x^3} = \sum_{n=0}^{\infty}(-1)^n x^{3n}.$$

We can then integrate this to get

$$\int_0^x \frac{1}{1+t^3}dt = \sum_{n=0}^{\infty}(-1)^n \frac{x^{3n+1}}{3n+1}.$$

This is allowed since we are within the radius of convergence of the power series. Putting $x = 0.5$ gives

$$I = \sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\cdot\frac{1}{2^{3n+1}} = \frac{1}{2} - \frac{1}{4\cdot 2^4} + \frac{1}{7\cdot 2^7} - \frac{1}{10\cdot 2^{10}} + \dots$$

Let a_n be the positive part of the alternating series, i.e.

$$a_n = \frac{1}{(3n+1)2^{3n+1}}.$$

Now, recall how to estimate alternating series: an alternating series starting at N is estimated in absolute value by a_N. Thus we need to compute the coefficients a_n up to an n such that $a_n < 10^{-3}$. We do this by hand and get

$$\begin{aligned} a_0 &= \frac{1}{2} = 0.5, \\ a_1 &= \frac{1}{64} \approx 0.0156, \\ a_2 &= \frac{1}{7 \cdot 2^7} \approx 0.001116, \\ a_3 &= \frac{1}{10 \cdot 2^{10}} \approx 0.000098 < 10^{-3}. \end{aligned}$$

If you have trouble with this it might be useful to learn how to calculate magnitudes of numbers quickly (e.g. for a_3 we can notice that $2^{10} \approx 10^3$ so that $a_3 \approx 10^{-4} < 10^{-3}$, so it takes only seconds to realise that a_3 is of the required precision). It follows by (1.12) that

$$|I - (a_0 - a_1 + a_2)| \leq a_3 < 10^{-3},$$

that is I is approximated to 3 decimal digits by the first 3 terms in the alternating series. Hence,

$$I \approx a_0 - a_1 + a_2 \approx 0.485.$$

1.10 Taylor Series

Recommended problems from §11.10: **11, 12, 15, 17, 18, 32, 44**.

So far we've observed that power series represent functions that have very strong properties. That is, within their radius of convergence, they are infinitely differentiable with easy-to-find derivatives and integrals (just differentiate/integrate term by term), they also make it easy to approximate the corresponding function by truncating the series. Two natural questions remain:

A Which functions have a power series expansion?

B Can a function have more than one power series about the same point?

We'll first answer question **B**. Suppose a function f has a power series expansion

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

First we note that if we set $x = c$ then all but the constant coefficient on the RHS disappear:

$$f(c) = a_0.$$

Hence the first coefficient is completely determined by f (so for a fixed f the value of a_0 is unique). We can then differentiate f and obtain

$$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \dots$$

and thus

$$f'(c) = a_1,$$

so the same is true for the a_1 coefficient. Let's keep differentiating.

$$f''(x) = 2a_2 + 3 \cdot 2a_3(x-c) + \dots$$

so that

$$f''(c) = 2a_2,$$

i.e.

$$a_2 = \frac{f''(c)}{2}.$$

Continuing in this manner yields for the k-th derivative

$$f^{(k)}(x) = k!a_k + (k+1)!a_{k+1}(x-c) + \dots$$

and again

$$f^{(k)}(c) = k!a_k.$$

We hence arrive at the formula

$$\boxed{a_k = \frac{f^{(k)}(c)}{k!}} \tag{1.16}$$

for all k. It follows that each coefficient in the power series of f centered at c is determined uniquely by f. Thus we conclude that a function can only have one power series about any given point. We can substitute this formula back into the coefficients of the power series of f to obtain the expression

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n. \tag{1.17}$$

This is called the **Taylor series** expansion of f at the point c. For $c = 0$ we sometimes call the resulting power series **Maclaurin series**, i.e.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

So when does an f have such an expansion? This is our question **A**. In order to answer it we have to refer to a famous theorem.

Taylor's Theorem. *Suppose f is $n+1$-times differentiable at c. Then it has the (finite) expansion*

$$f(x) = f(c) + f'(c)(x-c) + \ldots + \frac{1}{n!}f^{(n)}(c)(x-c)^n + R_n,$$

for some remainder R_n. *The function minus the remainder is called the* ***Taylor polynomial*** *of degree n, which we denote by P_n, that is*

$$P_n = \sum_{k=0}^{n} \frac{f^{(k)(c)}}{k!}(x-c)^k.$$

Here it is understood that both P_n and R_n depend on x.

So in conclusion we can write f as

$$f(x) = P_n + R_n.$$

We then say that *if* the limit of the remainder is 0, i.e.

$$R_n \to 0 \quad \text{as } n \to \infty,$$

then f converges to its Taylor series at c given by (1.17). Unfortunately we don't know exactly what the remainder is, but what we do know is that it has (many) convenient expressions that are easy to estimate. In this course we'll only see one form for the remainder known as **Lagrange's Remainder**:

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}, \tag{1.18}$$

for some ξ between x and c, that is,

$$\text{if } x > c, \text{ then } c < \xi < x,$$
$$\text{if } x < c, \text{ then } x < \xi < c.$$

It follows that ξ is bounded from above by $\max(x, c)$ and from below by $\min(x, c)$. It is important to realise that ξ depends on both x and n (c is fixed throughout).

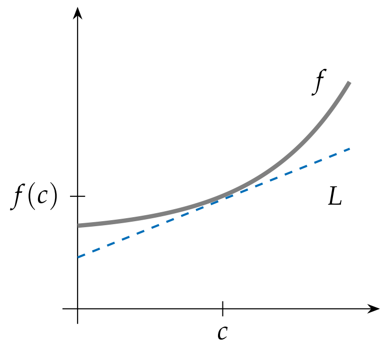


Figure 1.5: Linear approximation of a function.

The idea behind Taylor's Theorem is exactly the same as what you've already encountered when using the first derivative of a function to obtain linear approximations. Consider the formula for a derivative.[6] For x close to c it gives

$$f'(c) \approx \frac{f(c) - f(x)}{c - x}.$$

Rearranging we get

$$f(x) \approx f(c) + f'(c)(x - c). \tag{1.19}$$

Thus we see that the tangent line L given by the RHS of (1.19) is a good approximation to f near the point $x = c$ (see Figure 1.5). This is because both f and L have the same first derivative at c. Suppose now that we require that f and our approximation agree for even higher order derivatives. Our approximation then gets better and better around $x = c$. If we assume that all the possible derivatives (infinitely many) agree, then we automatically arrive at the Taylor expansion (1.17) (you should check that both sides of this equation have equal k-th derivatives for any k). Next we'll see how this works in practice.

Example. Find Taylor series for $f(x) = e^x$ (sometimes we write this function as $\exp x$) at 0.

To find the expansion we need to compute the n-th derivative of f. Clearly, $f'(x) = e^x$ and thus $f^{(k)}(x) = e^x$. It follows that

$$f(0) = f'(0) = \ldots = f^{(k)}(0) = 1.$$

By Taylor's theorem we then have

$$e^x = \sum_{k=0}^{n} \frac{1}{k!} x^k + R_n,$$

where the remainder is

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1},$$

[6] This way of writing it down might be unusual to you. Don't worry, you can get from the more common form of a derivative to this by a simple change of variables.

for some ξ between x and 0. We need to prove that $R_n \to 0$ as $n \to \infty$. We'll make use of the following fact

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0, \tag{1.20}$$

for a fixed x. This follows from the proof that the power series for e^x has a radius of convergence ∞ (because then the general term tends to 0).[7] Now, we can bound the remainder as

$$|R_n| = \left| \frac{1}{(n+1)!} x^{n+1} e^{\xi} \right| \le \frac{|x|^{n+1}}{(n+1)!} e^{\max(x,0)} \to 0$$

by (1.20) and the fact that x is fixed. We have thus showed that the exponential function converges to its Taylor series

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n,$$

for all $x \in \mathbb{R}$.

Similarly we can obtain the following Taylor series which converge for all $x \in \mathbb{R}$:

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n} = 1 - \frac{1}{2} x^2 + \frac{1}{4!} x^4 - \ldots,$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1} = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \ldots,$$

and

$$\cosh x = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n},$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}.$$

Then we also have the series we've seen before

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n},$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1},$$

which converge for $|x| < 1$ (in fact the series for arctan also converges for $x = \pm 1$, show this!). In Figures 1.6–1.8 we see how the Taylor polynomial of degree n converges to the corresponding function as n increases. Notice that this convergence only applies within the radius of convergence and beyond this the behaviour of the polynomials is erratic (in particular in Figure 1.8 where the radius of convergence is 1). In myCourses you can find the version of these pictures which was shown in the lectures.

Another useful expansion is the following. Let's first recall the usual Binomial Theorem. Let $n \in \mathbb{N}$, then the Binomial Theorem says that

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = 1 + nx + \frac{n(n-1)}{2} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \ldots + x^n,$$

[7] This is a bit backwards proof, because we're only now proving that e^x is equal to this power series. However, if you think about it it makes no difference what the power series is equal to, just that it converges for all x. It's possible to prove that the limit is 0 directly as well. You should try this.

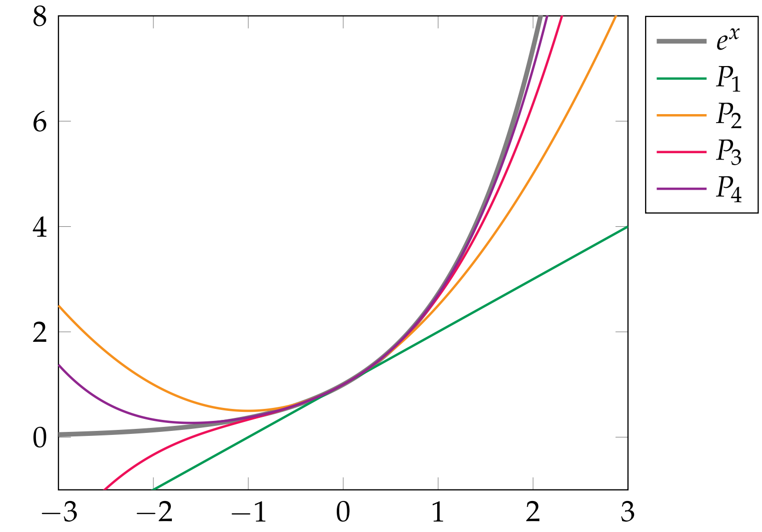


Figure 1.6: e^x and some Taylor Polynomials.

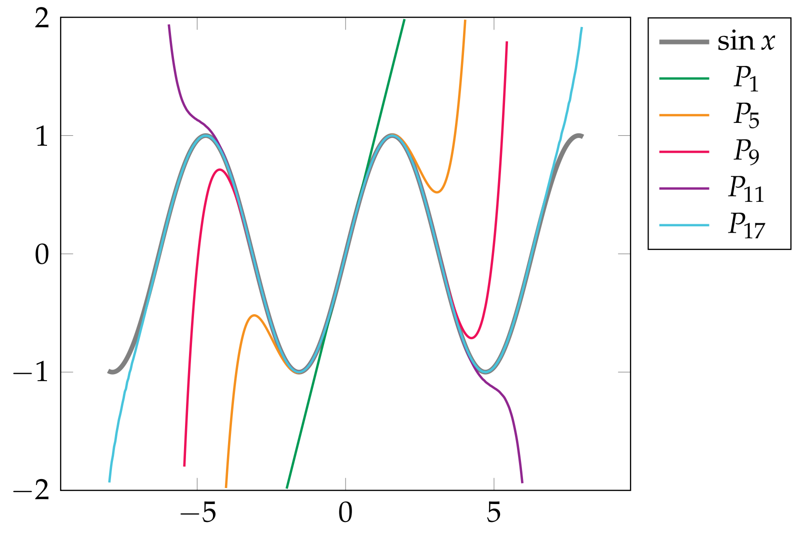


Figure 1.7: $\sin x$ and some Taylor Polynomials.

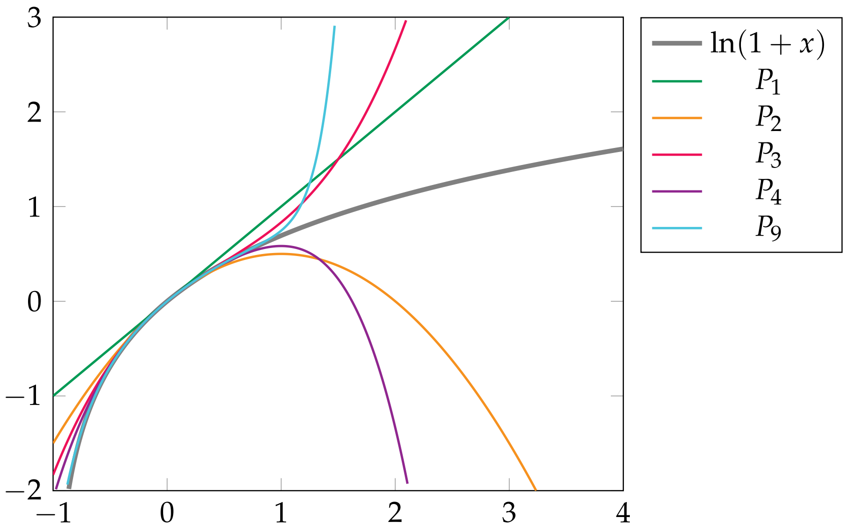


Figure 1.8: $\ln(1+x)$ and some Taylor Polynomials.

where the binomial coefficients can be obtained through the formula

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

Sometimes $\binom{n}{k}$ is denoted by ${}^nC_k$ (read: n choose k), because it represents the number of ways of choosing k items out of n elements (with replacement).

What happens if n is not an integer? Let $f(x) = (1+x)^n$. We can try to find the Taylor series for this by computing the k-th derivative at 0. We see that

$$\begin{aligned}
f(x) &= (1+x)^n & f(0) &= 1, \\
f'(x) &= n(1+x)^{n-1} & f'(0) &= n, \\
f''(x) &= n(n-1)(1+x)^{n-2} & f''(0) &= n(n-1), \\
&\vdots & &\vdots \\
f^{(k)}(x) &= n(n-1)\cdots(n-k+1)(1+x)^{n-k} & f^{(k)}(0) &= n(n-1)\cdots(n-k+1).
\end{aligned}$$

Hence we see that if f has a power series (at 0) then it is of the form

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots = \sum_{k=0}^{\infty} \binom{n}{k} x^k, \qquad (1.21)$$

where (by abusing the notation) $\binom{n}{k}$ denote the coefficients in this expansion (called the *generalised binomial coefficients*). To be rigorous we should prove that the remainder $R_k \to 0$ for the binomial expansion, but we'll just take it as a fact. Let's find the radius of convergence of (1.21). We have

$$\left| \frac{\binom{n}{k+1} x^{k+1}}{\binom{n}{k} x^k} \right| = |x| \left| \frac{n(n-1)\cdots(n-k+1)(n-k)}{n(n-1)\cdots(n-k+1)} \cdot \frac{k!}{(k+1)!} \right| = |x| \frac{|n-k|}{k+1} \to |x|,$$

as $k \to \infty$, since n is fixed (notice that in the book they interchange the letters k and n, I prefer to stick with this more standard notation). Hence the radius of convergence of (1.21) is 1 (so it is valid for $|x| < 1$). For some values of k the series converges at the end points, but it is simpler just to check it by hand when needed. Notice also that from (1.21) it follows that if n is a positive integer then eventually (for $k > n$) the coefficients are all 0, so we get a finite sum. That is, we can deduce the usual Binomial Theorem from this more general formula.

Most often Taylor's Theorem is not that useful in practice since it can be quite tedious to compute the n-th derivative of a complicated function. Instead we want to use our knowledge of common Taylor expansions and the tricks we've seen to compute Taylor series more quickly.

Example. Let

$$f(x) = x^3 \sin x^2.$$

Find P_{20} and the value of $f^{(17)}(0)$. You *don't* want to differentiate 20 times to find the Taylor polynomial! Instead we use the series expansion for sine:

$$\begin{aligned} f(x) &= x^3 \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} \\ &= x^3 \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots \right) \\ &= x^5 - \frac{x^9}{3!} + \frac{x^{13}}{5!} - \frac{x^{17}}{7!} + \dots \end{aligned}$$

This is enough to obtain P_{20} as the next term in the series has degree 21 so it is not part of P_{20}. Thus

$$P_{20}(x) = x^5 - \frac{x^9}{3!} + \frac{x^{13}}{5!} - \frac{x^{17}}{7!}.$$

Also, by the formula (1.16) we see that

$$f^{(17)}(0) = a_{17} \cdot 17! = \frac{-17!}{7!} = 8 \cdot 9 \cdots 16 \cdot 17 = \prod_{i=8}^{17} i.$$

We'll now work through lots of different examples involving Taylor series.

Example. Compute the limit

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2}=L.$$

This is of the type $0/0$ so you could try to apply L'Hôpital's Rule. Here's a different approach. Use the power series for e^x to get

$$\begin{aligned}
L &= \lim_{x\to 0}\frac{\left(1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\dots\right)-1-x}{x^2}\\
&= \lim_{x\to 0}\frac{\frac{x^2}{2}+\frac{x^3}{3!}+\dots}{x^2}\\
&= \lim_{x\to 0}\left(\frac{1}{2}+\frac{x}{3!}+\dots\right)\\
&= \frac{1}{2}.
\end{aligned}$$

To be precise, here we can take the limit of the series term by term because the power series converges uniformly (independently of x, roughly speaking). This always holds within the interval of convergence. We won't prove it in this course.

Example. Find the Taylor polynomial of degree 5 (about 0) of

$$f(x)=3x^2+2x+x^3.$$

But the Taylor polynomial is just f itself (i.e. $P_5=f$). Make sure you understand why! What is the degree 10 Taylor polynomial of f (at 0)? What about degree 2?

Example. Again, let

$$f(x)=3x^2+2x+x^3.$$

Find the Taylor series of f at 1.

First notice that since f is a polynomial this will be a finite sum. Now, to find the expansion we could just manipulate the terms to the form $((x-1)+1)$ and expand. But sometimes it's easier to define a new variable. Let $u=x-1$. We then want to expand f in terms of u. We have $x=u+1$ and hence

$$\begin{aligned}
f(u+1) &= 3(u+1)^2+2(u+1)+(u+1)^3\\
&= 3(u^2+2u+1)+2(u+1)+(u^3+3u^2+3u+1)\\
&= 6+11u+6u^2+u^3.
\end{aligned}$$

If we now substitute back for x we get

$$f(x)=6+11(x-1)+6(x-1)^2+(x-1)^3,$$

which is the Taylor series of f at 1.

Example. Find the power series of $\sin x$ at $x = \frac{\pi}{4}$. By computing a bunch of derivatives of $\sin x$ we see that there's a pattern:

$$f(x) = \sin x, \qquad f'(x) = \cos x,$$
$$f''(x) = -\sin x, \qquad f'''(x) = -\cos x,$$

and in particular $f^{(4)}(x) = \sin x = f(x)$, so we get back to where we started from. Hence we can write the n-th derivative of sine as

$$\frac{d^n}{dx^n}\sin x\Big|_{x=\frac{\pi}{4}} = \begin{cases} \sin\frac{\pi}{4} & \text{if } n \equiv 0 \pmod 4, \\ \cos\frac{\pi}{4} & \text{if } n \equiv 1 \pmod 4, \\ -\sin\frac{\pi}{4} & \text{if } n \equiv 2 \pmod 4, \\ -\cos\frac{\pi}{4} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Here $n \equiv a \pmod 4$ means that n divided by 4 leaves a remainder a (read: n is congruent to a modulo 4). Since $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ (see Figure 1.9), we arrive at

$$\sin x = \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(x - \tfrac{\pi}{4}\right)^n,$$

where

$$a_n = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } n \equiv 0, 1 \pmod 4, \\ \frac{-1}{\sqrt{2}} & \text{if } n \equiv 2, 3 \pmod 4. \end{cases}$$

We can write the series more explicitly as

$$\sin x = \frac{1}{\sqrt{2}}\left(1 + u - \frac{u^2}{2} - \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} - \dots\right),$$

where $u = x - \frac{\pi}{4}$.

Alternatively, we could try to do the same trick as in the previous examples, and just forcefully write the argument of $\sin x$ in terms of $x - \frac{\pi}{4}$. This leads to (after a trigonometric identity)

$$\begin{aligned} \sin x &= \sin((x - \tfrac{\pi}{4}) + \tfrac{\pi}{4}) \\ &= \sin(u + \tfrac{\pi}{4}) \\ &= \sin u \cos\tfrac{\pi}{4} + \cos u \sin\tfrac{\pi}{4} \\ &= \tfrac{1}{\sqrt{2}}(\sin u + \cos u), \end{aligned}$$

and then we can just use the power series for sin and cos to get the same result.

Let's do a similar example, but with the Binomial series.

Example. Find the degree 2 Taylor polynomial for

$$f(x) = (23 - 3x)^{2/3},$$

at $x = 5$. Since this is a carefully chosen example, let's try to manipulate the term inside to contain $(x-5)$ (otherwise it might be better to go with the substitution trick used in the previous examples). This yields

$$\begin{aligned} f(x) &= (23 - 3((x-5)+5))^{2/3} \\ &= (8 - 3(x-5))^{2/3}. \end{aligned}$$

Now to find the expansion we want to use the Binomial Theorem (1.21), but to do that we need the first term to be 1. Therefore we factor out the 8 to get

$$f(x) = 8^{2/3}(1 - \tfrac{3}{8}(x-5))^{2/3}.$$

Then applying the Binomial Theorem yields

$$f(x) = 4(1 + (\tfrac{2}{3})(\tfrac{-3}{8})(x-5) + \tfrac{1}{2}(\tfrac{2}{3})(\tfrac{2}{3}-1)(\tfrac{-3}{8})^2(x-5)^2 + \ldots$$

We can then simplify to see that

$$P_2 = 4 - (x-5) - \frac{1}{16}(x-5)^2.$$

We'll next see how to use power series to deduce cool identities.

Example. Let

$$S = \sum_{n=1}^{\infty} \frac{3^{-n}}{n}.$$

Find the value of S.

To do questions like this we first try to identify if the series in question reminds us of any power series that we know. Here S resembles the power series of $\ln(1+x)$ except that we're missing the $(-1)^{n+1}$ term. As we see, this can be taken care of:

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}.$$

Let $x = -y$, then

$$\begin{aligned} \ln(1-y) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-y)^n}{n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}y^n}{n} \\ &= -\sum_{n=1}^{\infty} \frac{y^n}{n}. \end{aligned}$$

And thus putting $y = 1/3$ yields

$$S = -\ln(1 - \tfrac{1}{3}) = -\ln\tfrac{2}{3} = \ln\tfrac{3}{2},$$

by using the property $a \ln b = \ln b^a$.

Example. Find the value of the sum

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}.$$

This sum should make you think of the power series for cos since after you bring the π and $1/6$ together the general term is of the form $(-1)^n x^{2n}/(2n)!$. Indeed, setting $x = \pi/6$ in the power series for cos yields

$$\cos\frac{\pi}{6} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\pi}{6}\right)^{2n}.$$

What's left is to figure out what is the value of $\cos\frac{\pi}{6}$. For this it's useful to recall the help triangles:

Figure 1.9: Help triangles (not to scale).

With these (and the help of unit circle and a few trig identities) it's possible to figure out values for the most common expressions of sine and cosine you'll encounter. You should memorise these (if you haven't already)! From the triangle on the left we see immediately that $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and hence we've shown that

$$\frac{\sqrt{3}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}.$$

So we've managed to obtain an algebraic closed form for a power series involving π. Neat, huh? Next we'll see how to obtain an identity for π from a power series of a rational expression.

Example. Show that

$$\pi = 2\sqrt{3}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}.$$

The terms here are of the form $(-1)^n x^n/(2n+1)$ so this series is a bit reminiscent of the series for arctan that we found in lecture 7. Recall we proved that

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}. \tag{1.22}$$

So how do we get what we want? Well, if we think of the 3^n in the denominator as $(\sqrt{3})^{2n}$ then, save for a factor of $\sqrt{3}$, the denominator is taken care of. Now recall from the help triangles (Figure 1.9) that

$$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}.$$

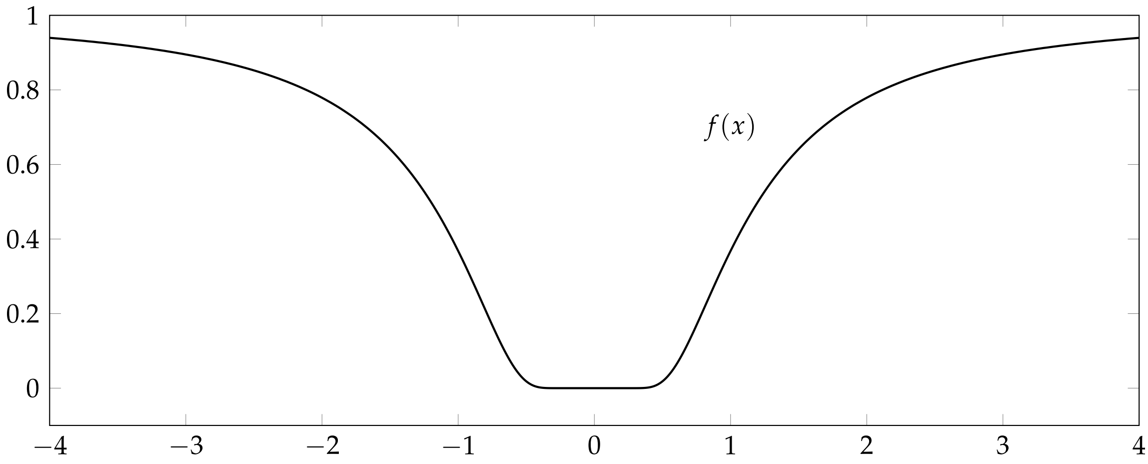


Figure 1.10: Plot of $f(x) = \exp(-1/x^2)$.

Then, setting $x = 1/\sqrt{3}$ in (1.22) we get

$$\frac{\pi}{6} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}}.$$

Simplifying a bit gives

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}.$$

Finally, multiplying both sides by 6 gives the result we wanted.

Example. Now we take a look at a function which is very "nice" (infinitely differentiable everywhere), but only converges to its Taylor series at a single point. Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}.$$

We notice that as $x \to \pm\infty$ then $f(x) \to 1$ so we can graph this function as in Figure 1.10. So we see that the function is bounded and it is also infinitely differentiable everywhere. This is clear for $x \neq 0$, but at 0 we have to be a bit more careful. We can compute the first derivative as

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(h+0) - f(0)}{h - 0} \\ &= \lim_{h \to 0} \frac{e^{-1/h^2}}{h}, \end{aligned}$$

which we can rewrite as

$$\begin{aligned} &= \lim_{h \to 0} \frac{1/h}{e^{1/h^2}} \\ &= \lim_{h \to 0} \frac{-1/h^2}{(-1/h^3)e^{1/h^2}}, \end{aligned}$$

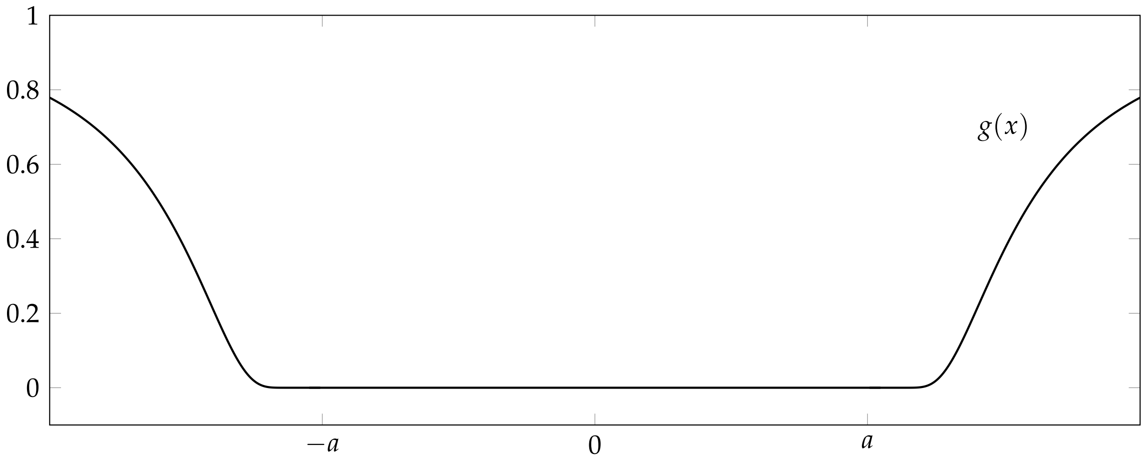


Figure 1.11: Plot of a generic flat function.

by L'Hôpital's Rule. The remaining limit is easy to evaluate as

$$= \lim_{h \to 0} \frac{h}{e^{1/h^2}} = 0.$$

It is possible (but very tedious) to show with similar argument that the n-th derivative of f at 0 is 0, i.e. $f^{(n)}(0) = 0$. But then it follows that the Taylor expansion of f at 0 is just

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots = 0 + 0 + 0 + \ldots$$

Hence, since $\exp(-1/x^2) > 0$ for any $x \in \mathbb{R}$ it follows that f only agrees with its Taylor expansion at a single point, that is, at $x = 0$.

Why does this happen? On paper, this should be a perfect function to use Taylor series for, especially since the Taylor series has radius of convergence ∞ (since all the terms are 0).[8] From Figure 1.10 we can begin to see where the problem lies. The function has so strong decay (exponential) at the origin that locally around 0 it looks almost completely flat. This kind of behaviour becomes impossible to approximate with polynomials, which is why the Taylor expansion fails to yield anything useful. Notice that with a simple modification we can make the flat part of f as long as we want. Define

$$g(x) = \begin{cases} e^{-1/(x-a)^2} & \text{if } x > a, \\ 0 & \text{if } |x| \leq a, \\ e^{-1/(x+a)^2} & \text{if } x < -a, \end{cases}$$

for some $a > 0$. Then this function has a flat part of length $2a$, see Figure 1.11.

The previous example highlights the important point that a function is not always equal to its Taylor series, even if the corresponding Taylor series converges. The precise reason here is that if you try to compute the remainder in Taylor's theorem and look at

[8] In particular, this means that to prove that a function has a Taylor expansion one cannot naively just compute what the Taylor series should be and calculate its radius of convergence.

its limit (recall, this is how we proved the existence of Taylor series for functions), then it does not tend to 0 (unless $x = 0$). The deeper reason is that the function $\exp(-1/x^2)$ has a very strong singularity at 0. Let's write formally (not rigorously)

$$e^{-1/x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{-2n}}{n!} = 1 - \frac{1}{x^2} + \frac{1}{2!\,x^4} - \frac{1}{3!\,x^6} + \dots$$

This is, of course, not a Taylor series (since it has negative exponents of x). Instead we can consider the same function in terms of a complex variable $z \in \mathbb{C}$. That is

$$e^{-1/z^2} = 1 - \frac{1}{z^2} + \frac{1}{2!\,z^4} - \frac{1}{3!\,z^6} + \dots = \sum_{n=-\infty}^{0} \frac{(-1)^n z^{2n}}{n!}.$$

This is what we call a *Laurent series* of a function (since it has negative powers of z), and this particular expansion has an *essential singularity* at 0 since the negative exponents go up to $-\infty$. As a function of a complex variable, it means that as $z \to 0$ then e^{-1/z^2} assumes every complex value (except the value 0) infinitely often! A similar, but less drastic, behaviour can be observed with e.g. the function $h(x) = 1/(1 + x^2)$. This is well-defined for all $x \in \mathbb{R}$ and even infinitely differentiable everywhere so it should provide another good example of a well-behaved Taylor series. But this is not so, can you figure out why? What is the radius of convergence of the Taylor series of h at 0? You will again need the fact that if a power series (at 0) has a radius of convergence R then it not only needs to converge for all $|x| < R$, but also for any $|z| < R$, where $z \in \mathbb{C}$ (Hint: what is the Laurent expansion of h?). In any case, power series with complex variables are not part of this course so this is just for your own interest.

Example. Let
$$f(x) = \frac{1}{1+x+x^2+x^3}.$$
We want to find the Taylor expansion of f at 0.

There are various ways of working this out. The naive way is just to substitute in to geometric series formula, i.e.
$$\frac{1}{1+(x+x^2+x^3)} = 1 - (x+x^2+x^3) + (x+x^2+x^3)^2 - \dots$$
and then simplify. But this is quite tedious.

A better approach is to notice that (since $x = -1$ is clearly a root) the denominator factorises as
$$1+x+x^2+x^3 = (1+x)(1+x^2).$$
Hence we can use partial fractions to write f as
$$\begin{aligned} f(x) &= \frac{1}{(1+x)(1+x^2)} \\ &= \frac{A}{1+x} + \frac{Bx+C}{1+x^2}, \end{aligned}$$
and solve for A, B and C, to get
$$f(x) = \frac{1/2}{1+x} + \frac{(1-x)/2}{1+x^2}.$$
Now it's a lot easier to apply geometric series to get that
$$f(x) = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n x^n + \frac{1-x}{2}\sum_{n=0}^{\infty}(-1)^n x^{2n},$$
which you can simplify easily.

An even quicker approach would be to notice that
$$(1-x)(1+x+x^2+x^3) = 1-x^4,$$
from which it follows that we can write f as
$$f(x) = \frac{1-x}{1-x^4}.$$
It is then a simple matter of expanding the single geometric series to get
$$f(x) = (1-x)(1+x^4+x^8+x^{12}+\dots) = 1 - x + x^4 - x^5 + x^8 - x^9 + \dots$$
This factorisation is a specific case of a general formula that I call *Fact of Life*
$$\boxed{\alpha^n - \beta^n = (\alpha-\beta)(\alpha^{n-1} + \alpha^{n-2}\beta + \dots + \alpha\beta^{n-2} + \beta^{n-1}).}$$
So to get our example we just set $\alpha = 1$, $\beta = x$ and $n = 4$.

Example. Here's a question from a past midterm: decide if the series

$$\sum_{n=1}^{\infty} \frac{\tan \frac{1}{n}}{\sqrt{n}} = \sum_{n=1}^{\infty} a_n$$

converges.

There are a couple of ways to approach this problem. First we do a quick check (as one always should, at least mentally) of whether the general term tends to 0. Since $\tan 0 = 0$ and tan is continuous at 0, it follows that $a_n \to 0$, as required. So there is no easy way out.

So we know that $\tan \frac{1}{n} \to 0$ as $n \to \infty$. But the question is: at what rate? If it decays faster than $1/\sqrt{n}$ then our series should converge (by comparison with a p-series with $p > 1$). How do we figure out what p to use?

In order to determine the rate of decay we can use the Taylor expansion of $\tan x$. It's only necessary to compute the first few terms so it's possible to do it by differentiation. But this is a bit time-consuming and error-prone, which is not ideal in an exam. A better way is to use the definition of tan as a ratio of sin and cos and divide the power series:

$$\tan x = \frac{\sin x}{\cos x}.$$

We multiply by $\cos x$ and compare coefficients. Notice that $\tan x$ is odd so its power series expansion only has odd coefficients (similarly an even function only has even coefficients in its Taylor expansion about 0), thus we get

$$\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots\right)\left(b_1 x + b_3 x^3 + b_5 x^5 + \dots\right) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

For the first coefficient we immediately see that $b_1 = 1$. Then, for x^3 we observe

$$b_3 - \frac{b_1}{2} = \frac{-1}{3!},$$

so that

$$b_3 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}.$$

Next,

$$b_5 - \frac{b_3}{2} + \frac{a_1}{4!} = \frac{1}{5!},$$

which rearranges to

$$b_5 = \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} = \frac{2}{15}.$$

Thus

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots \tag{1.23}$$

Hence, plugging in $x = \frac{1}{n}$ gives

$$\tan \frac{1}{n} = \frac{1}{n} + \frac{1}{3n^3} + \frac{2}{15n^5} + \dots. \tag{1.24}$$

So that each term decays to 0. The decay is dominated by the slowest rate, which is clearly $1/n$, thus we see that as $n \to \infty$ then

$$\tan \frac{1}{n} \approx \frac{1}{n}.$$

Hence we should try to compare our series with $\sum_n \frac{1}{\sqrt{n}} \frac{1}{n} = \sum_n n^{-3/2}$. For the limit comparison test we get

$$\left| \frac{\tan \frac{1}{n}}{\sqrt{n}} n^{3/2} \right| = n \tan \frac{1}{n} \to 1,$$

by the Taylor expansion (1.24).

An even quicker way is to just remember the fact that $\sin x \approx x$ for small x and that $\cos x \to 1$ as $x \to 0$, thus

$$\tan x \approx x$$

for small x, and then we can proceed as before. Then to evaluate the limit in the limit comparison test, we can just use L'Hôpital's Rule instead of having to compute the Taylor expansion:

$$\lim_{n\to\infty} \frac{\tan \frac{1}{n}}{1/n} = \lim_{n\to\infty} \frac{\frac{-1}{n^2} \sec^2 \frac{1}{n}}{-1/n^2} = \sec^2 0 = 1 > 0,$$

since sec is continuous at 0. Thus we see again that the series $\sum_n a_n$ converges by the limit comparison test with $\sum_n n^{-3/2}$.

Example. Here's another question from an old midterm. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n}(4x+3)^n.$$

Find the interval of convergence of this series and the function it represents in this interval explicitly.

We begin by computing the radius of convergence. We have

$$\left| \frac{(4x+3)^{n+1} n}{(n+1)(4x+3)^n} \right| = |4x+3| \frac{n}{n+1} \to |4x+3|,$$

as $n \to \infty$. Hence by the ratio test we need $|4x+3| < 1$, so $|x + 3/4| < 1/4$ and the radius of convergence is $1/4$. More precisely, we've shown that the series converges for

$$\frac{-1}{4} < x + \frac{3}{4} < \frac{1}{4},$$
$$-1 < x < \frac{-1}{2}.$$

We still need to check what happens at the end points. For $x = -1$ our series becomes

$$f(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges by the alternating series test. On the other hand, for $x = \frac{-1}{2}$ we just get the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges. Thus the interval of convergence is

$$[-1, \tfrac{-1}{2}).$$

Now, to find the sum of the series we have a couple of ways to work it out. We could immediately recognise that the series for f is of the form $\sum_n y^n/n$, which resembles the series for the logarithm. In particular, if we recall that

$$\ln(1+y) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n},$$

(which converges for $|y| < 1$) then setting $y = -(4x+3)$ (which is allowed within the interval of convergence, i.e. where $|4x+3| < 1$ and also at the left end point) gives

$$\ln(1-(4x+3)) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-(4x+3))^n}{n} = -\sum_{n=1}^{\infty} \frac{(4x+3)^n}{n} = -f(x),$$

so that

$$f(x) = -\ln(-4x-2).$$

What if we can't remember (or recognise that it is) the series for $\ln(1+x)$? We can also try to directly manipulate the series into something we know how to sum up. The series for f is of the type

$$\sum_{n=1}^{\infty} \frac{w^n}{n},$$

where we set $w = 4x+3$ for simplicity. Now we want to be able to write this in terms of a geometric series (which we know we can sum up). Since we have a factor n in the denominator, we know that it appears from integration. Hence we can write

$$\sum_{n=1}^{\infty} \frac{w^n}{n} = \int_0^w \sum_{n=1}^{\infty} u^{n-1}\, du,$$

which is clearly true since we can integrate term by term. By re-indexing (since the first term in the sum is u^0) we get

$$\sum_{n=1}^{\infty} \frac{w^n}{n} = \int_0^w \sum_{n=0}^{\infty} u^n\, du,$$

which is just a geometric series and thus

$$\sum_{n=1}^{\infty} \frac{w^n}{n} = \int_0^w \frac{1}{1-u} du = [-\ln(1-u)]_0^w = -\ln(1-w).$$

Substituting $w = 4x+3$ gives the same result as before.

An important point that I didn't emphasise during the lecture is that this is decidedly different from finding the Taylor expansion of a function around some point. That is, I can't find the Taylor expansion for $\ln(1+x)$ about some other point e.g. 4 by just substituting $(x-4)$ into the Taylor expansion of $\ln(1+x)$ about 0. Make sure you understand why (hint: look at the general formula for Taylor expansion. When you change x what else needs to change apart from the term $(x-c)^n$?)! On the other hand, when we are finding the sum of the series, like in this question, we are working within the interval of convergence, so if I use the series expansion of $\ln(1+y)$ then I can substitute for y any expression that is less than 1 in absolute value (like $4x+3$ in our example). It is important that you understand this distinction! In summary: the difference is that in this question we are tasked to find the sum of a specific series as opposed to expanding a function about some specific point.

Example. Find the sum and interval of convergence of

$$S = \sum_{n=0}^{\infty} (-1)^n (3^n - 2^n) x^n.$$

We again begin by finding the radius of convergence:

$$\left| \frac{(-1)^n \left(3^{n+1} - 2^{n+1}\right) x^{n+1}}{(-1)^n \left(3^n - 2^n\right) x^n} \right| = |x| \frac{3 - 2\left(\frac{2}{3}\right)^n}{1 - \left(\frac{2}{3}\right)^n} \to 3|x| < 1,$$

so the radius of convergence is $1/3$ and the series converges for at least $x \in (-1/3, 1/3)$. We still need to check the end points: for $x = \frac{1}{3}$ we get

$$\sum_{n=0}^{\infty} (-1)^n \frac{3^n - 2^n}{3^n} \sum_{n=0}^{\infty} (-1)^n \left(1 - \left(\tfrac{2}{3}\right)^n\right) = \sum_{n=0}^{\infty} a_n.$$

But then we notice that $a_{2n} \to 1$ while $a_{2n+1} \to -1$ so therefore a_n diverges and in particular $a_n \not\to 0$, and thus the series diverges for $x = \frac{1}{3}$. At the other end point, $x = \frac{-1}{3}$ we see even easier that the series

$$\sum_{n=0}^{\infty} (1 - (\tfrac{2}{3})^n),$$

diverges since the general term doesn't tend to 0. Therefore the interval of convergence is just

$$(-\tfrac{1}{3}, \tfrac{1}{3}).$$

Now to find the sum of the series, S, we simply split the sum to

$$S = \sum_{n=0}^{\infty} (-1)^n (3x)^n - \sum_{n=0}^{\infty} (-1)^n (2x)^n.$$

Notice that the radius of convergence of the first sum is $1/3$ and for the second sum it's $1/2$ so both of them certainly converge within the interval of convergence of S (i.e. $(-\frac{1}{3}, \frac{1}{3})$). Therefore we are allowed to split the series. Now we can just use the formula for geometric series to obtain

$$S = \frac{1}{1+3x} - \frac{1}{1+2x} = \frac{-x}{(1+3x)(1+2x)}.$$

Here's a couple of more examples of Taylor series, which I didn't do during the lecture. I might present some of them on Monday.

Example. Use Taylor's theorem to approximate $\sqrt[3]{8.2}$ to 5 decimal digits. What's the maximum error in this method?

For this we want to consider the function $f(x) = \sqrt[3]{x}$ and to find its Taylor expansion about some point c near $x = 8.2$ where we can compute $f(c)$ easily. A sensible choice is $c = 8$, then

$$\begin{aligned} f(x) &= \sqrt[3]{x} & f(8) &= 2, \\ f'(x) &= \frac{1}{3}x^{-2/3} & f'(8) &= \frac{1}{12}, \\ f''(x) &= \frac{-2}{9}x^{-5/3} & f''(8) &= \frac{-1}{144}, \\ f'''(x) &= \frac{10}{27}x^{-8/3}, \end{aligned}$$

and we'll see that these are enough for our approximation. We then compute the Taylor polynomial of f of degree 2 at 8:

$$\begin{aligned} P_2(x) &= f(8) + f'(8)(x-8) + \frac{f''(8)}{2!}(x-8)^2 \\ &= 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2, \end{aligned}$$

which, at $x = 8.2$, becomes just

$$P_2(8.2) = 2 + \frac{0.2}{12} - \frac{(0.2)^2}{288}.$$

This we can easily simplify (without calculator) to

$$\begin{aligned} P_2(8.2) &= 2 + \frac{1}{12} \cdot \frac{2}{10} - \frac{1}{288} \cdot \frac{4}{100} \\ &= 2 + \frac{1}{60} - \frac{1}{7200} \\ &\approx 2.01653 \end{aligned}$$

by rounding up. To prove that this is a valid approximation we use Taylor's theorem to write

$$\sqrt[3]{8.2} = P_2(8.2) + R_2.$$

It is then enough to show that $|R_2| < 10^{-5}$. By using the formula for the remainder we have

$$R_2 = \frac{f'''(\xi)}{3!}(8.2-8)^3,$$

where $8 < \xi < 8.2$. It follows that

$$2 < \xi^{1/3} < \sqrt[3]{8.2},$$

and thus

$$\xi^{-8/3} < 2^{-8}.$$

Therefore, substituting this into the formula for $f'''(\xi)$ we see that

$$f'''(\xi) \leq \frac{10}{27}\frac{1}{2^8} = \frac{5}{3^3 2^7}.$$

Notice that here the inequality is *sharp* (since ξ could be arbitrarily close to 8). It follows that the remainder is bounded by

$$|R_2| \leq \frac{5}{3^3 2^7} \cdot \frac{1}{3 \cdot 2} \left(\frac{2}{10}\right)^3 = \frac{1}{129\,600},$$

which is the maximal error in our approximation (since all the inequalities are sharp). This is easily seen to satisfy

$$|R_2| \approx 0.0000077 < 10^{-5},$$

so that our approximation is valid to 5 decimal digits (after rounding up).

Example. Let

$$f(x) = \sum_{n=0}^{\infty} \frac{(n+2)^2}{n+1} x^n.$$

Find the interval of convergence of the series and the value of the sum explicitly.

Start again by applying the ratio test to $a_n = \frac{(n+2)^2}{n+1} x^n$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+3)^n (n+1) x^{n+1}}{(n+2)(n+2)^2 x^n}\right| = \frac{(1+\frac{3}{n})^2(1+\frac{1}{n})}{(1+\frac{2}{n})^3}|x| \to |x|,$$

from which it follows that the radius of convergence is 1. What happens at the end points? Notice that

$$\frac{(n+2)^2}{(n+1)} > \frac{(n+1)^2}{n+1} = n+1 \to \infty$$

so that $x^n(n+2)^2/(n+1)$ diverges for both $x = \pm 1$. It follows that the interval of convergence is

$$(-1, 1).$$

To compute f explicitly we have to do a bit more work. A good starting point is to simplify the fraction as

$$\frac{(n+2)^2}{n+1} = \frac{n^2+4n+4}{n+1} = n+3+\frac{1}{n+1}. \tag{1.25}$$

To see this we simply do long division with polynomials:

$$\begin{array}{r@{}l}
 & n+3 \\
n+1\big) & \;n^2+4n+4 \\
 & -n^2-n \\
\hline
 & 3n+4 \\
 & -3n-3 \\
\hline
 & 1
\end{array}$$

Now to find the sum we could be clever and rewrite the series as

$$f(x) = \sum_{n=0}^{\infty}\left(n+3+\frac{1}{n+1}\right)x^n = \sum_{n=0}^{\infty}\left((n+1)+2+\frac{1}{n+1}\right)x^n,$$

since I recognise that $\sum_{n=0}(n+1)x^n$ is just the derivative of a geometric series as we've seen before. That is, I have

$$\sum_{n=0}^{\infty}(n+1)x^n = \frac{d}{dx}\sum_{n=0}^{\infty}x^{n+1} = \frac{d}{dx}\frac{x}{1-x} = \frac{1}{(1-x)^2}. \tag{1.26}$$

We also have easily that

$$\sum_{n=0}^{\infty}2x^n = \frac{2}{1-x}. \tag{1.27}$$

Finally we have to look at $\sum_{n=0} x^n/(n+1)$. We could immediately recognise that this is similar to the series for ln and write

$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}.$$

We can then try to manipulate this so that the series on the RHS will match exactly with the one for $\sum_{n=0} x^n/(n+1)$. First we set $x \mapsto -x$ which yields

$$\ln(1-x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-x)^n}{n} = -\sum_{n=1}^{\infty}\frac{x^n}{n}.$$

That is,

$$-\ln(1-x) = \sum_{n=1}^{\infty}\frac{x^n}{n} = x + \frac{x^2}{2} + \dots$$

Now if we divide both sides by x and reindex the sum we arrive at exactly what we wanted

$$\frac{-1}{x}\ln(1-x) = \sum_{n=1}^{\infty}\frac{x^{n-1}}{n} = \sum_{n=0}^{\infty}\frac{x^n}{n+1}. \tag{1.28}$$

Thus, putting (1.26), (1.27) and (1.28) together gives us that

$$f(x) = \frac{1}{(1-x)^2} + \frac{2}{1-x} - \frac{1}{x}\ln(1-x). \tag{1.29}$$

If you don't like manipulating the series directly we can also bruteforce our way through the computations with a more direct approach. Let's first suppose that I don't do the grouping $(n+1)+2+\frac{1}{n+1}$. Then I have to find

$$\sum_{n=0}^{\infty}\left(n+3+\frac{1}{n+1}\right)x^n.$$

Look at the series with nx^n first. Then, by using the derivative of a power series (again), we see that

$$\begin{aligned}\sum_{n=0}^{\infty}nx^n &= \sum_{n=1}^{\infty}nx^n && \text{(since the first term is just 0)}\\ &= x\sum_{n=1}^{\infty}nx^{n-1},\end{aligned}$$

and then by introducing the derivative

$$
\begin{aligned}
&= x\frac{d}{dx}\sum_{n=1}^{\infty} x^n \\
&= x\frac{d}{dx}\frac{1}{1-x} \\
&= \frac{x}{(1-x)^2}.
\end{aligned}
$$

As before we easily get

$$\sum_{n=0}^{\infty} 3x^n = \frac{3}{1-x}.$$

And for the last series we can use integration (since we have the $n+1$ factor in the denominator) to obtain the same result as before:

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{n+1}x^n &= \frac{1}{x}\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} \\
&= \frac{1}{x}\int_0^x \sum_{n=0}^{\infty} t^n\,dt \\
&= \frac{1}{x}\int_0^x \frac{1}{1-t}dt \\
&= \frac{1}{x}\left[-\ln(1-t)\right]_0^x = \frac{-1}{x}\ln(1-x).
\end{aligned}
$$

You should check that if you combine these three functions then you get exactly the same answer for f as we got in (1.29).

Notice that by setting, for example, $x = \frac{1}{2}$ we arrive at the identity

$$\sum_{n=0}^{\infty} \frac{(n+2)^2}{n+1}2^{-n} = f(\tfrac{1}{2}) = 8 + 2\ln 2.$$

So if you were asked to prove that this identity is true, then a good way to do it would be to realise that you can work more abstractly with the power series $f(x)$ and then set $x = 1/2$ at the end, as we've done.

Our last example emphasises the point that you need to keep careful track of the limits of summation.

Example. Let

$$S = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}3^{-n}. \tag{1.30}$$

Find the value of the sum S.

We again realise that we want to work with power series of the form

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{2n+1},$$

for some x. This is reminiscent of the power series for arctan which we saw is just

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1},$$

which upon setting $x \mapsto \sqrt{x}$ becomes what we have above with a crucial difference that the series start at different terms. Anyway, if you don't recognise the series for arctan then we can again bruteforce our way through it by employing the gometric series and some integration. First of all, to arrive at the series that I want, (1.30), I have to integrate something that gives me the $(-1)^n/(2n+1)$ factor. The easiest answer is to use the integral $\int_0^x (-1)^n t^{2n} dt = (-1)^n x^{2n+1}/(2n+1)$. Which series has a general term of the form $(-1)^n t^{2n}$? Well the geometric series of common ratio $-t^2$ of course:

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \ldots = \sum_{n=0}^{\infty} (-1)^n t^{2n}.$$

Integrating both sides from 0 to y gives

$$\int_0^y \frac{1}{1+t^2} dt = \int_0^y \sum_{n=0}^{\infty} (-1)^n t^{2n} dt.$$

And therefore by the derivative of $\arctan y$ we have

$$[\arctan t]_0^y = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{2n+1}.$$

It's now important to notice that the sum on the RHS is not quite what we want (if we were to set $y^2 = 1/3$), because the summation starts at $n = 0$ instead of $n = 1$ like in S. Thus we separate the $n = 0$ term to get

$$\arctan y - \arctan 0 = y + \sum_{n=1}^{\infty} \frac{(-1)^n y^{2n+1}}{2n+1},$$

so that

$$\arctan y - y = \sum_{n=1}^{\infty} \frac{(-1)^n y^{2n+1}}{2n+1}.$$

Dividing by y gets us where we wanted:

$$\frac{\arctan y}{y} - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n y^{2n}}{2n+1},$$

which is exactly S for $y^2 = 1/3$, i.e. $y = 1/\sqrt{3}$. Therefore

$$S = \frac{\arctan \frac{1}{\sqrt{3}}}{1/\sqrt{3}} - 1 = \sqrt{3}\frac{\pi}{6} - 1.$$

It is again useful to recall the help triangles (Figure 1.9) we saw before to determine the value of $\arctan \frac{1}{\sqrt{3}}$.

Chapter 2

Vectors and Three-dimensional Space

Before moving on to calculus in higher dimensions, we give a brief recap of basic concepts on vectors and three-dimensional space.

2.1 3D Space

Recommended problems from §12.1: **15, 17, 44**.

Our setting will be the three dimensional real vector space

$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}.$$

That is, for a point $P \in \mathbb{R}^3$, we define it by a triple $P = (a, b, c)$. An important quantity is the distance of P from the origin $O = (0, 0, 0)$, let's denote it by d (see Figure 2.1). To compute this we first consider the **projection** of P onto the xy-plane, say, $P' = (a, b, 0)$. Then on the xy-plane we have the right triangle with catheti a and b. Thus the

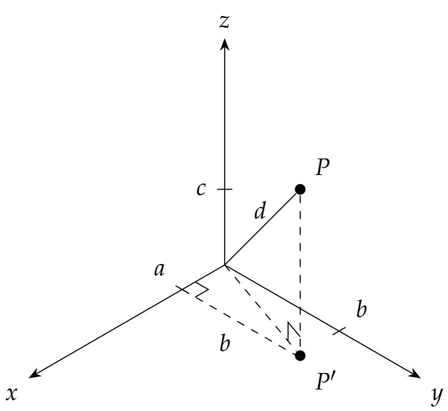


Figure 2.1: A right-handed three-dimensional coordinate system.

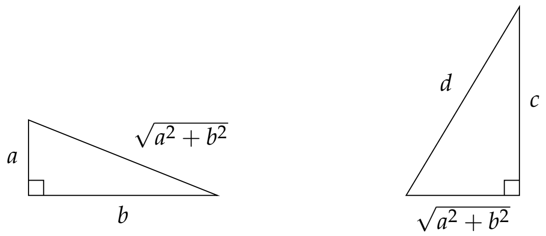


Figure 2.2: Triangles from Figure 2.1.

hypotenuse is $\sqrt{a^2+b^2}$. But this is then the base of the other right triangle with height c and hypotenuse d (see Figure 2.2). It follows that $d = \sqrt{a^2+b^2+c^2}$. We state this as a definition.

Definition 2.1. For a point $P = (a, b, c) \in \mathbb{R}^3$ its **distance** from the origin O is

$$|P| = \sqrt{a^2+b^2+c^2}.$$

More generally, if $Q = (a', b', c')$ is another point in $\mathbb{R}^3$ then the distance between P and Q is

$$|P-Q| = \sqrt{(a-a')^2+(b-b')^2+(c-c')^2}.$$

Now, recall that in two dimensions if we have a function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = y$, then this defines a curve on the plane $\mathbb{R}^2$ given by

$$\{(x, f(x)) : x \in \mathbb{R}\},$$

which we call the *graph of f*. Similarly, for $g : \mathbb{R}^2 \to \mathbb{R}$ with $g(x, y) = z$, this defines a $3-1 = 2$-dimensional object (we are fixing one variable)

$$\{(x, y, g(x, y)) : x, y \in \mathbb{R}\},$$

called the *surface defined by g*. Later we'll see how to define surfaces by using vectors, but for now we'll take a look at a few examples of surfaces defined by equations.

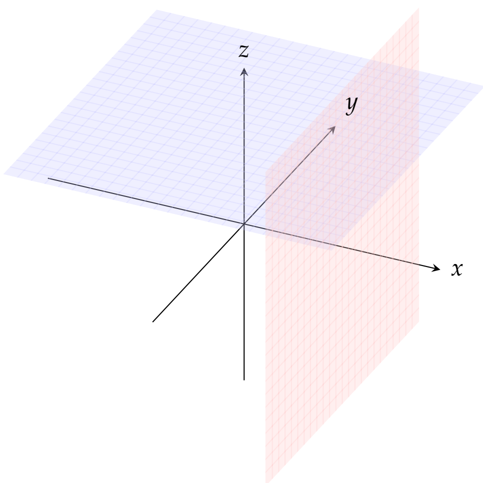


Figure 2.3: The planes $z = 3$ and $x = 3$ in blue and red, respectively.

Example. If we look at the equation $z = 3$, then this means that (with the above notation) the function $g(x, y)$ is a constant. Thus the set of points the equation defines is just

$$\{(x, y, 3) : x, y \in \mathbb{R}\}.$$

This is a plane (see Figure 2.3).

An important point to emphasises here is that the equation $x = 5$, for example, defines a different set of points depending on whether you are in $\mathbb{R}^2$ or $\mathbb{R}^3$. In two dimensions it is just a line, whereas in three dimensions it is a plane. Therefore it is important to keep the context of where you are working in clear in your head.

Example. Let's look at the equation

$$x^2 + y^2 = 1.$$

In two dimensions this equation gives the unit circle, whereas in three dimensions we have a new free variable z, so we can extend the circle infinitely in the positive and negative z direction. Hence we arrive at a cylinder (see Figure 2.4).

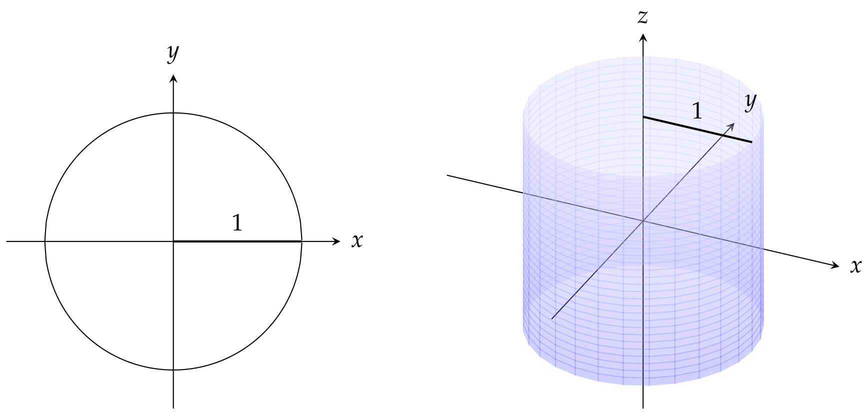


Figure 2.4: The equation $x^2 + y^2 = 1$ in 2D and 3D.

What about the equation $x^2 + z^2 = 2$? This also defines a cylinder, but in a different direction as we see in Figure 2.5.

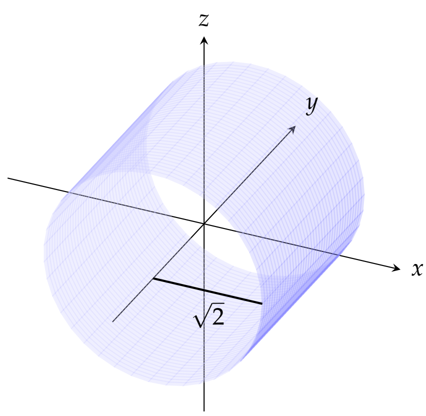


Figure 2.5: The cylinder given by the equation $x^2 + z^2 = 2$.

Example. Now let's take
$$x^2 + y^2 + z^2 = 1.$$
Clearly this just defines the set of all points whose distance squared (from the origin) is 1. But since distance is always positive it follows that this set is the same as the set of points with *distance* 1 from the origin. The locus of these points is of course the unit sphere.

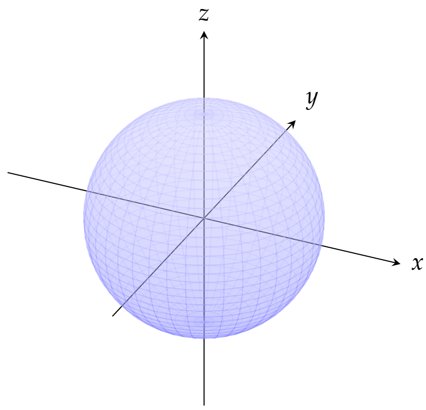


Figure 2.6: The unit sphere given by the equation $x^2 + y^2 + z^2 = 1$.

In general,
$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$
describes a sphere of radius r centred at (a, b, c).

Example. What kind of a surface is described by the equation
$$x^2 + y^2 + z^2 + 8x - 6y + 2z + 17 = 0?$$
This is not immediately obvious. To see the answer we complete the square in each of the variables. This yields
$$(x+4)^2 - 16 + (y-3)^2 - 9 + (z-1)^2 - 1 + 17 = 0,$$
which simplifies to
$$(x+4)^2 + (y-3)^2 + (z-1)^2 = 3^2.$$
This is a sphere of radius 3 centred at $(-4, 3, 1)$.

Example. The next example is not a surface, but a *curve* in three-dimensional space. In particular, the set of equations
$$x = y = 3$$
define a line instead of a surface since we have 2 equations in 3 dimensional space, which means that there is only one degree of freedom left (compare this to the case of a cylinder, for example).

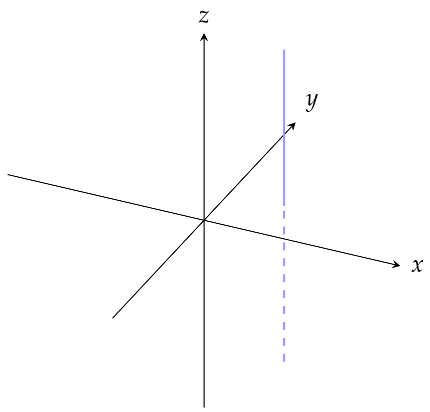


Figure 2.7: A line given by the equations $x = y = 3$.

2.2 Vectors

Vectors provide us an useful abstraction that can be used to describe many objects in higher dimensional space. In other words, many formulas which use vectors stay identical from one dimension to another. Hence, for the rest of this section we'll mostly be stating any details in the case of $\mathbb{R}^3$, but the same statements apply (with obvious modifications) to $\mathbb{R}^2$. A separate definition will be provided wherever confusion might arise.

A **vector** is an object that has both a *direction* and a *magnitude* (length). In particular, a vector does not have a specified starting point. This means that any two vectors with the same direction and length are equal, no matter where in the space they are located. More concretely, this means that we can move vectors around without changing anything. Therefore, without loss of generality (wlog) we can consider all vectors as starting from the origin O. In other words, to any point $P \in \mathbb{R}^3$, we can associate a unique vector which starts from O and ends at P. We denote this vector by $\overrightarrow{OP}$. Then the vector $-\overrightarrow{OP}$ is just the same vector as $\overrightarrow{OP}$ but with the opposite direction (that is $\overrightarrow{PO} = -\overrightarrow{OP}$).

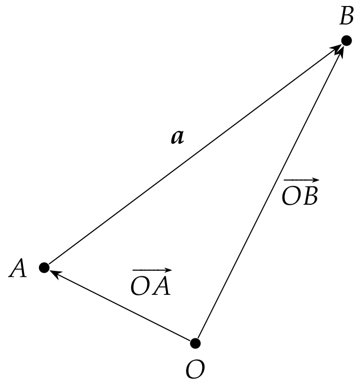


Figure 2.8: Vector from A to B. Here $\boldsymbol{a} = \overrightarrow{AB}$.

Let's take two points $A = (2, 2, 1)$ and $B = (3, 2, 0)$ in $\mathbb{R}^3$. Then the vector $\overrightarrow{AB}$ has to be a vector that takes us from the point A to point B. From Figure 2.8 it is clear that $\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$ as we first travel from A to O and then from O to B. For our particular choice of A and B we get (by using the matrix notation)

$$\overrightarrow{AB} = \begin{pmatrix} 3-2 \\ 2-2 \\ 0-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

The values 1, 0 and -1 are called the **components** of the vector $\overrightarrow{AB}$. You might have seen vectors being denoted by square brackets before, it is the same thing (I also use round brackets for all matrices). Finally, the vector $\mathbf{0} = \left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right)$ is the **zero vector**, which is the only vector with no direction. From now on we will mostly use bold typeface letters to denote vectors, e.g. $\overrightarrow{AB} = \boldsymbol{a}$ in Figure 2.8. On a blackboard we use underlined letters, e.g. $\underline{a}$. You might've used overlines before, it is the same thing.

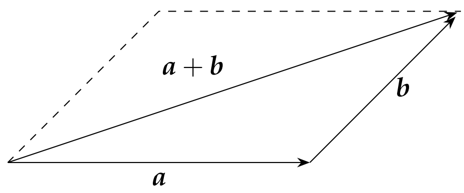


Figure 2.9: Addition of two vectors.

We'll next state some basic definitions regarding vector operations.

Definition 2.2. Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ be vectors. From now on, unless otherwise specified, we write the components of these vectors as $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. Also let $\alpha \in \mathbb{R}$. Then we define **vector addition** as the vector

$$\boldsymbol{a} + \boldsymbol{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}.$$

Pictorially we have the following situation: two non-parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ define a *parallelogram* (by moving $\boldsymbol{b}$ to start from the tip of $\boldsymbol{a}$, recall we can do this) whose diagonal is $\boldsymbol{a} + \boldsymbol{b}$ as we show in Figure 2.9. Another basic operation is **scalar multiplication** of $\boldsymbol{a}$ by the **scalar** $\alpha \in \mathbb{R}$. This is a vector given by

$$\alpha\boldsymbol{a} = \begin{pmatrix} \alpha a_1 \\ \alpha a_2 \\ \alpha a_3 \end{pmatrix}.$$

Notice that the vector $\alpha\boldsymbol{a}$ is a vector in the same (or opposite) direction as $\boldsymbol{a}$. This is why we sometimes also call this operation **scaling**. Moreover, for two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ we say that $\boldsymbol{a}$ is a **scalar multiple** of $\boldsymbol{b}$ if we can find a constant $c \in \mathbb{R}$ such that

$$\boldsymbol{a} = c\boldsymbol{b}.$$

If this is the case then $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel. Sometimes we denote this by $\boldsymbol{a} \parallel \boldsymbol{b}$. Finally, by recalling the formula for distance in $\mathbb{R}^3$, we define the length of a vector as the length of the line segment from the origin to the unique choice of point in $\mathbb{R}^3$ defined by that vector, i.e.

$$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^3}.$$

Notice that $\boldsymbol{a}$ and $-\boldsymbol{a}$ are parallel. Here are some basic properties of the above operations.

Proposition 2.3. *Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ be vectors and $\alpha, \beta \in \mathbb{R}$ be scalars. Then*

(i)
$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a} \qquad \text{(commutativity)}$$

(ii)
$$\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} \qquad \text{(associativity)}$$

(iii)
$$a + \mathbf{0} = a \qquad \text{(additive identity)}$$

(iv)
$$a + (-a) = \mathbf{0} \qquad \text{(additive inverse)}$$

(v)
$$\alpha(a + b) = \alpha a + \alpha b \qquad \text{(distributive laws)}$$

(vi)
$$(\alpha + \beta)a = \alpha a + \beta b$$

(vii)
$$(\alpha\beta)a = \alpha(\beta a) \quad \text{(compatibility of scalar multiplication and multiplication in } \mathbb{R})$$

(viii)
$$1 \cdot a = a \qquad \text{(identity for scalar multiplication)}$$

If you are familiar with group theory then you can notice that these rules imply that the set of vectors of a vector space is an abelian group under vector addition. We single out an useful set of vectors.

Definition 2.4. In $\mathbb{R}^n$ the set of vectors

$$\left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\},$$

where e_i has a 1 in the i-th component and zeros elsewhere, is called the **standard basis** for $\mathbb{R}^n$.

In $\mathbb{R}^3$ we usually use the notation

$$i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

for the standard basis vectors (and analogously for $\mathbb{R}^2$). That is, these are vectors of length 1 that point in the directions of our coordinate system (that is along the x, y and z-axes) as shown in Figure 2.10. From your linear algebra class you might remember that a basis for a vector space means that it is possible to write every non-zero vector in the vector space uniquely in terms of the basis vectors. For us this is not too important in general. However, in $\mathbb{R}^3$ this means that all vectors can be written in terms of i, j and k. After a little bit of thought you'll see that the way to do this is

$$a = a_1 i + a_2 j + a_3 k.$$

It's important to notice that all the vectors i, j and k have length 1. We call such vectors unit vectors.

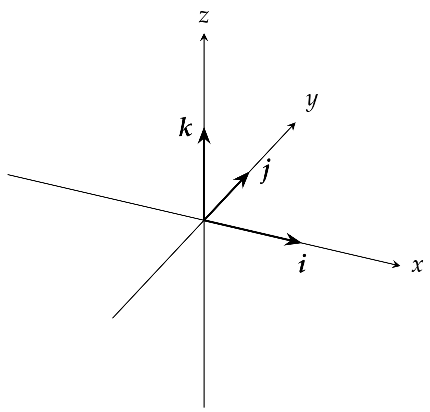


Figure 2.10: The standard basis in $\mathbb{R}^3$.

Definition 2.5. A vector $\boldsymbol{a}$ of length 1 is called a **unit vector**. A unit vector in the direction of $\boldsymbol{a}$ is denoted by $\hat{\boldsymbol{a}}$.

For any non-zero vector $\boldsymbol{a}$ we can easily write the unit vector in the direction of $\boldsymbol{a}$ by the following formula

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}, \tag{2.1}$$

that is we just multiply $\boldsymbol{a}$ by the inverse of its magnitude. You should check by using the definition of the length of a vector that $|\hat{\boldsymbol{a}}| = 1$. Also, to be really precise we should be using $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$, but for the sake of brevity and clarity of notation we won't. Next we'll look at the two standard ways of multiplying vectors.

2.3 Dot Product

Recommended problems from §12.3: **25, 29, 31, 55.**

Also sometimes called a *scalar product*, this is the easiest way to multiply two vectors. It is especially useful in physical calculations, such as finding how much of a force is applied to a given direction (in mathematics we call this projecting).

Definition 2.6. For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, their **dot product** is given by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

In particular, the result of this product is a *scalar* quantity (hence the terminology *scalar product*). It is also possible to give an equivalent but more geometric definition of the dot product. This turns out to be

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta, \tag{2.2}$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. In general in 3 dimensions you can understand the angle between two vectors as the angle between the line segments represented by

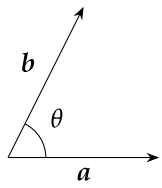


Figure 2.11: The angle θ between two vectors.

the vectors when we move them to start from the origin. However, in practice it's easier to just use (2.2) as the definition of the angle between two vectors. Here are some basic properties of the dot product. Basically the gist of it is that everything works as you expect it to.

Proposition 2.7. *Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be vectors and let $\alpha \in \mathbb{R}$ be a scalar. Then*

(i)
$$\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$$

(ii)
$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$$

(iii)
$$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$$

(iv)
$$(\alpha \boldsymbol{a}) \cdot \boldsymbol{b} = \alpha(\boldsymbol{a} \cdot \boldsymbol{b}) = \boldsymbol{a} \cdot (\alpha \boldsymbol{b})$$

(v)
$$\boldsymbol{0} \cdot \boldsymbol{a} = 0.$$

Another useful observation is that it follows from (2.2) that if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ then $\cos\theta = 0$ so $\theta = \pm\frac{\pi}{2}$ (for $\boldsymbol{a}, \boldsymbol{b} \neq \boldsymbol{0}$). This means that $\boldsymbol{a}$ and $\boldsymbol{b}$ are *perpendicular* (*orthogonal*), which we denote by $\boldsymbol{a} \perp \boldsymbol{b}$.

As we stated before, a dot product is particularly useful for finding the components of a vector in a given direction (loosely speaking, how much the vector points in some direction). Observe that by basic trigonometry the length of the base of the triangle (with base in direction $\hat{\boldsymbol{a}}$) and hypotenuse $\boldsymbol{b}$ is just $|\boldsymbol{b}|\cos\theta$ (see Figure 2.12).

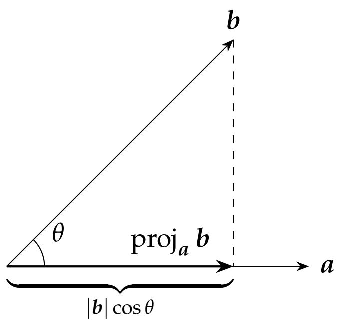


Figure 2.12: Projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$.

The quantity $|\boldsymbol{b}|\cos\theta$ is then called the **scalar projection** of $\boldsymbol{b}$ onto $\boldsymbol{a}$. We can turn this into a vector by taking the unit vector in the direction of $\boldsymbol{a}$ and multiplying it with the scalar projection. Thus we arrive at

$$\operatorname{proj}_{\boldsymbol{a}} \boldsymbol{b} = |\boldsymbol{b}|\cos\theta\, \hat{\boldsymbol{a}} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}\,\hat{\boldsymbol{a}},$$

which is called the **vector projection** of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$.

Finally, we state another way of expressing the unit vector in a given direction, but it won't be as useful to us as (2.1). First, denote the angles a vector $\boldsymbol{a}$ makes with the vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ (alternatively, with the x, y and z-axes) by α, β and γ, respectively. These are called the **direction angles** of $\boldsymbol{a}$. Furthermore, by taking the cosine of these we get e.g.

$$\cos\alpha = \frac{\boldsymbol{a}\cdot\boldsymbol{i}}{|\boldsymbol{a}||\boldsymbol{i}|} = \frac{a_1}{|\boldsymbol{a}|},$$

and similarly for β and γ. These are called the **direction cosines** of $\boldsymbol{a}$. It is then immediate that

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|} = \begin{pmatrix}\cos\alpha\\ \cos\beta\\ \cos\gamma\end{pmatrix}.$$

2.4 Cross Product

Recommended problems from §12.4: **35, 44**.

The discussion in this section only applies to $\mathbb{R}^3$. It is possible to extend analogous ideas to other dimensions through something called the exterior product, but we won't delve into that here. We use a less explicit and a more geometric way of defining the cross product (or vector product) of two vectors.

Definition 2.8. For two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ their **cross product**, denoted by $\boldsymbol{a}\times\boldsymbol{b}$, is defined to be the unique vector that is

a) perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ with the orientation that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a}\times\boldsymbol{b}$ form a right-handed set of vectors (i.e. according to the right-hand rule we learnt);

b) of length equal to the area of the parallelogram defined by $\boldsymbol{a}$ and $\boldsymbol{b}$ (see e.g. Figure 2.9).

Here the property *a*) defines the *direction* and property *b*) the *magnitude* of the cross product vector. Recall that this is all we need to define a vector. It turns out that there is a nice formula to compute the components of the cross product: it can be expressed as a determinant of a 3×3 matrix. We find that

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} = \begin{pmatrix}a_2b_3 - a_3b_2\\ a_3b_1 - a_1b_3\\ a_1b_2 - a_2b_1\end{pmatrix}.$$

It follows immediately from Definition 2.8 and properties of the dot product that

$$\boldsymbol{a} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = 0.$$

Also, with a little bit of trigonometry, we get the following theorem from *b*).

Theorem 2.9. *Let $\theta \in [0, \pi]$ be the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. Then*

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin \theta. \tag{2.3}$$

Furthermore, we then see that if the cross product is zero then for non-zero vectors this must mean that $\sin \theta = 0$ so that $\theta = 0, \pi$, which means that $\boldsymbol{a}$ and $\boldsymbol{b}$ point either in the same or opposite directions.

Corollary 2.10. *Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$.*

We next list some basic properties of the cross product. The most important one is the first one which states that the cross product is not commutative (you've seen other non-commutative products before, such as matrix multiplication), but anti-commutative (i.e. one has to swap the sign).

Proposition 2.11. *Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}$ be vectors and $\alpha \in \mathbb{R}$ a scalar. Then*

(i)
$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$$

(ii)
$$(\alpha \boldsymbol{a}) \times \boldsymbol{b} = \alpha(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a} \times (\alpha \boldsymbol{b})$$

(iii)
$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$$

(iv)
$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$$

(v)
$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$$

The last two properties are collectively called the *triple product identities* (*scalar* and *vector* triple products, respectively). For the scalar triple product there is only one way to make sense of the product so the parentheses are not necessary, but for the vector triple product they do make a difference (since vector product is anti-commutative).

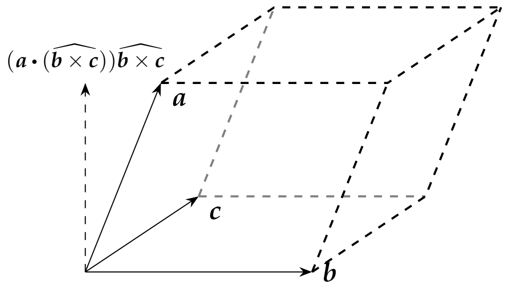


Figure 2.13: Volume of a parallelepiped.

Also, the scalar triple product is useful for the following geometric reason. Take three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ which do not all lie on the same plane (we say they are not *coplanar*). Then look at the parallelepiped defined by the vectors, with base given by $\boldsymbol{b}$ and $\boldsymbol{c}$, say (see Figure 2.13). The height of this parallelepiped is then the *projection* of $\boldsymbol{a}$ in the direction perpendicular to both $\boldsymbol{b}$ and $\boldsymbol{c}$ (which is of course the unit vector in the direction of $\boldsymbol{b} \times \boldsymbol{c}$). Since the area of the base is just the magnitude of $\boldsymbol{b} \times \boldsymbol{c}$ (by definition), it follows that the volume V of the parallelepiped is given by

$$V = |\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})|,$$

since the volume has to be positive.

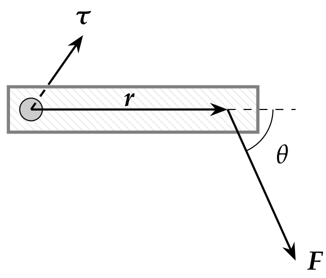


Figure 2.14: The torque on a lever as a cross product.

Before moving on to the next section we look at one physical application of the vector product. Suppose we have a lever fixed at one end (call it O) and free to rotate. If we then apply a force $\boldsymbol{F}$ to the lever at a point $\boldsymbol{r}$ (relative to O) along it, the lever would rotate (as long as we don't apply the force parallel to the lever). In physics this is called *torque* and the precise definition is

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}.$$

We see that torque is a vector, so what is the meaning of its direction and magnitude? Well, if we consider $\boldsymbol{\tau}$ as a vector from O then its direction gives the axis of rotation. On the other hand the magnitude $|\boldsymbol{\tau}| = |\boldsymbol{r} \times \boldsymbol{F}|$ is roughly speaking a measure of how much we are turning the lever. We show the configuration in Figure 2.14. Some important things to note: by using the formula (2.3) we see that

$$|\boldsymbol{\tau}| = |\boldsymbol{r}||\boldsymbol{F}| \sin\theta,$$

where θ is the angle between the lever and $\boldsymbol{F}$. Then if $\boldsymbol{\tau} = \boldsymbol{0}$ it follows that $\theta = 0$ or π (for non-zero force). So we get the expected physical behaviour that if we apply force along the lever then there will be no rotation. A maximal rotation is obtained when $\sin\theta = 1$, this means that $\theta = \frac{\pi}{2}$, i.e. when we apply the force perpendicular to the lever. A more interesting observation is that increasing the lever arm (the magnitude of $\boldsymbol{r}$) increases the torque! This is useful in real life. You can try for yourself by trying to open a door by first pushing close to the hinges and then by pushing closer to the handle. Which is easier?

2.5 Lines and Planes

Recommended problems from §12.5: **27, 33, 45, 73**.

To specify a line uniquely in three dimensions we need exactly the same information as in two dimensions: two separate points. So suppose we have points $P, Q \in \mathbb{R}^3$ and call the line passing through these points L. We can then form the vector $\overrightarrow{PQ}$ which points in the direction of the line. So if we want to arrive at any point on L from the origin we can first travel to the line through the vector $\overrightarrow{OP}$ and then along the line by scaling the

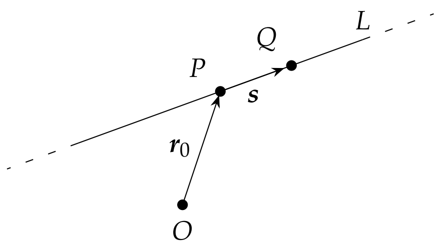


Figure 2.15: Parametrisation of a line.

vector $\overrightarrow{PQ}$. This is illustrated in Figure 2.15. If we denote an abritrary point on the line by $\boldsymbol{r}$, then the above discussion just translates to the equation

$$\boldsymbol{r} = \overrightarrow{OP} + \lambda\overrightarrow{PQ},$$

where λ is just the scaling parameter that was mentioned before. It is more convenient to write this in the standard vector notation. So denote the vector $\overrightarrow{OP}$ by $\boldsymbol{r}_0$ and the vector $\overrightarrow{PQ}$ by $\boldsymbol{s}$. Then we have

$$\boldsymbol{r} = \boldsymbol{r}_0 + \lambda\boldsymbol{s}, \tag{2.4}$$

which is called the **vector equation of a line**. So to summarise, here $\boldsymbol{r}$ is the position vector of an arbitrary point on the line (dependent on the parameter λ), $\boldsymbol{r}_0$ is the position vector of any given point on the line, and $\boldsymbol{s}$ is any vector that points in the direction of the line. If we let $\boldsymbol{r} = (x, y, z)^\intercal$ ($^\intercal$ just denotes the matrix transpose, that is swapping rows and columns. In other words you can just think of this as an equivalent way of writing a vector which is a bit cleaner to do inline), $\boldsymbol{r}_0 = (x_0, y_0, z_0)^\intercal$ and $\boldsymbol{s} = (a, b, c)^\intercal$, then (2.4) just becomes

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 + \lambda a \\ y_0 + \lambda b \\ z_0 + \lambda c \end{pmatrix}.$$

By equating the components of the vectors we arrive at the system

$$\begin{cases} x = x_0 + \lambda a \\ y = y_0 + \lambda b \\ z = z_0 + \lambda c \end{cases}.$$

These are the **parametric equations of a line**. Assuming a, b and c are non-zero we can solve these equations of λ and equate to get

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \quad (= \lambda).$$

These are the **symmetric equations of a line**. It is important to notice that if given a line in this form then it's always possible to obtain a vector in the direction of the line simply by composing a vector out of the denominators a, b and c.

Example. Before we had the equations $x = y = 3$, which we saw define a line. Notice that this is not in the standard forms of equations of a line. To put it in one of these forms we need a point on the line and a vector in the direction of a line. In this case it's easy to see that $\begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$ is a point on the line. It should also be clear that $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is a vector in the direction of the line. Thus we can write the vector equation of this line as

$$\boldsymbol{r} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

or in the parametric form

$$\begin{cases} x = 3 \\ y = 3 \\ z = \lambda \end{cases}.$$

Since both a and b (by using the same notation as before) are 0 it is impossible to write this line by using the symmetric equations of a line (instead we have to resort to writing it as $x = y = 3$ as stated).

To finish off our discussion on lines we take a look at how to parametrise a segment of a line. In particular given points with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (sometimes we might just simplify the exposition by just saying "*given points* $\boldsymbol{a}$ *and* $\boldsymbol{b}$" and refer to the vectors directly; this has the same meaning), the line segment from $\boldsymbol{a}$ to $\boldsymbol{b}$ can be written as

$$\boldsymbol{r}(t) = \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a}),$$

for $t \in [0, 1]$. You should understand this coming from the fact that our starting point $\boldsymbol{a}$ and then we want to move in the direction of the vector $\boldsymbol{b} - \boldsymbol{a}$. Then we just figure out the correct range for the parameter t. We can rewrite this as

$$\boldsymbol{r}(t) = (1 - t)\boldsymbol{a} + t\boldsymbol{b},$$

where $t \in [0, 1]$ again.

Next we'll see how to describe planes in 3 dimensions. To do this we use the following idea: given a vector that is normal to the plane (denoted by $\boldsymbol{n}$) then any vector lying on the plane has to be perpendicular to $\boldsymbol{n}$. Conversely, if we take any vector that is perpendicular to $\boldsymbol{n}$, then it necessarily has to lie on the plane (after possibly moving its starting point; recall that we can do this to vectors).

Let's be a bit more precise. Denote our plane by $\mathcal{P}$. The normal vector is still $\boldsymbol{n}$. Also let $\boldsymbol{r}$ be the position vector of an arbitrary point on the plane and let $\boldsymbol{r}_0$ be a given point on the plane. Then $\boldsymbol{r} - \boldsymbol{r}_0$ is a vector on the plane so it has to be orthogonal to $\boldsymbol{n}$ as we see in Figure 2.16. Thus we arrive at the **vector equation of a plane**

$$\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}_0) = 0.$$

If we let $\boldsymbol{n} = (a, b, c)^\intercal$ and $\boldsymbol{r}_0 = (x_0, y_0, z_0)^\intercal$, then we can simplify the above equation to (by expanding the dot product)

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

which is the **scalar equation of a plane**. We can further simplify this by collecting all the constant together to get

$$ax + by + cz = d. \tag{2.5}$$

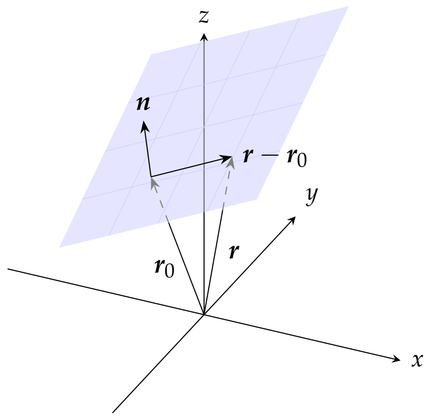


Figure 2.16: Vector definition of a plane.

Notice that this is slightly different from the equation given in the book (the d is on the other side of the equation, in equations this means we get the opposite sign in front of d). An important point to notice is that given a plane with an equation (2.5), we can immediately read off (without doing any computations) a vector which is normal to the plane just by combining the coefficients a, b and c in a single vector, $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Finally, recall that for a line we saw that we needed to specify two points on it (this is the same in all dimensions!). For a plane we'll notice that three points is enough, as long as they do not all lie on the same line (we say they are not *collinear*). Indeed, suppose P, Q and R are points on a plane $\mathcal{P}$. Then $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are non-parallel vectors on the plane and therefore we can form the cross product $\overrightarrow{PQ} \times \overrightarrow{PR}$. The cross product then gives us a normal vector to the plane which we can use as before to write the plane as

$$(\overrightarrow{PQ} \times \overrightarrow{PR}) \bullet (\boldsymbol{r} - \overrightarrow{OP}) = 0.$$

Example. For example, let $P = (1,1,1)$, $Q = (2,2,1)$ and $R = (1,2,2)$. Then a normal vector to the plane defined by these points is

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Thus the vector equation of the plane is

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix} = 0,$$

which we can rewrite as

$$(x-1) - (y-1) + (z-1) = 0,$$

or

$$x - y + z = 1.$$

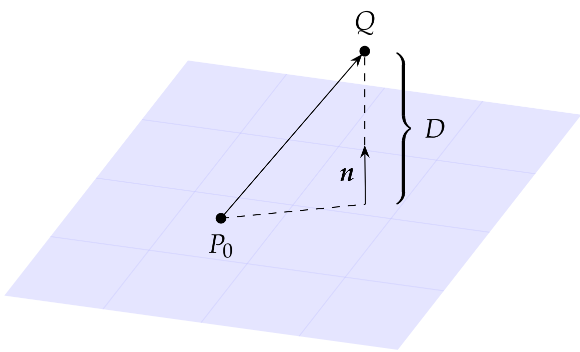


Figure 2.17: Distance of a point from a plane.

A final useful calculation for us is a formula for the distance between a point and a plane. Suppose we want to compute the distance between a point Q and a plane $\mathcal{P}$. To do this we pick any point P_0 from the plane and consider the vector $\overrightarrow{P_0Q}$, if we then project this vector onto the normal vector then (up to multiplication by the length of the normal vector) this projection gives the distance, as can be seen from Figure 2.17. We can express this idea in the form of an equation

$$D = \frac{|\overrightarrow{P_0Q} \cdot \boldsymbol{n}|}{|\boldsymbol{n}|}. \tag{2.6}$$

If we set $\mathcal{P} : ax + by + cz = d$, $Q = (x_1, y_1, z_1)$ and $P_0 = (x_0, y_0, z_0)$. Then we can substitute in (2.6) to get

$$\begin{aligned} D &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}. \end{aligned} \tag{2.7}$$

Notice that I had a typo in the lectures here: I wrote $+d$ instead of $-d$.

2.6 Examples of Surfaces in $\mathbb{R}^3$

Finally, we'll take a look at some further examples of surfaces in three dimensions.

Example.

$$\boxed{z = x^2}$$

To see what this equation represents we can simply notice that since the y variable does not appear in the equation at all it is "free". Hence, if we look at the plot of the equation on the xz-plane,

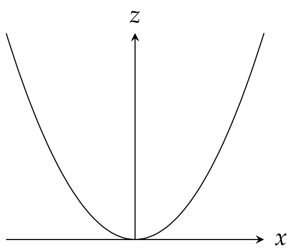


Figure 2.18: $z = x^2$ on the xz-plane.

then to see what the equation represents in three dimensions we just extend the curve infinitely in the negative and positive y direction. Thus we arrive at

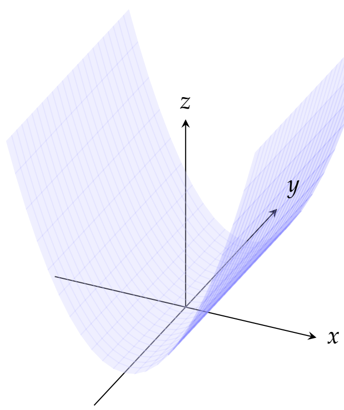


Figure 2.19: The parabolic cylinder $z = x^2$.

Surfaces of this type (one linear variable in terms of another quadratic one, with one variable missing) are called *parabolic cylinders*.

Example.

$$\boxed{x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1}$$

We can use the same technique as before and set $z = 0$ to figure out what the surface looks like on the xy-plane. For $z = 0$ we have $x^2 + \frac{y^2}{4} = 1$, which is an ellipse.

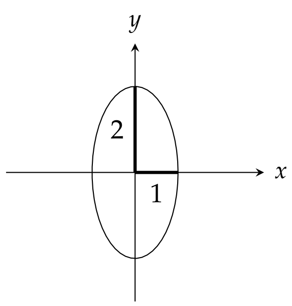


Figure 2.20: $x^2 + \frac{y^2}{4} = 1$ on the xy-plane.

On the xz-plane we get similarly (by setting $y = 0$) the equation $x^2 + \frac{z^2}{9} = 1$, so another ellipse.

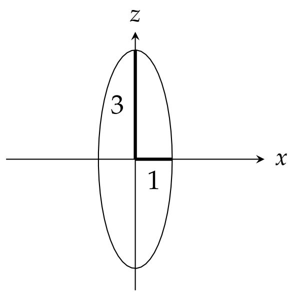


Figure 2.21: $x^2 + \frac{z^2}{9} = 1$ on the xz-plane.

In the 3D-space these look like the following:

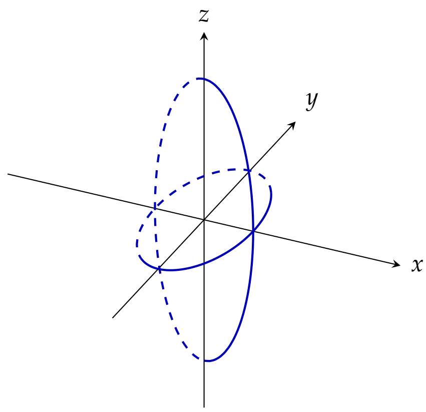


Figure 2.22: The previous curves in 3D-space.

With the help of these (and possibly a few more cross-cuts) it is not difficult to see that we arrive at a surface that looks like the shell of an egg.

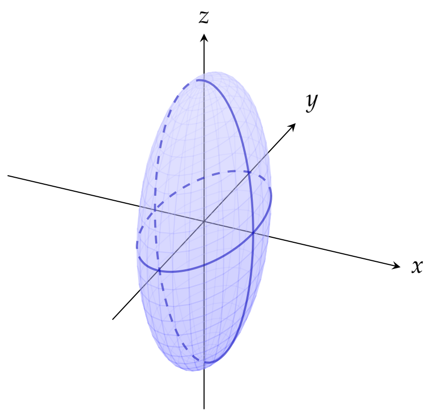


Figure 2.23: Curves on the surface of an ellipsoid.

This is an example of an *ellipsoid*. In general these are given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

This ellipsoid has the principal semi-axes a, b, c (i.e. the distance from the centre to the shell along each of the axes).

Example.

$$\boxed{z = 4x^2 + y^2}$$

In order to plot this surface we notice that $z \geq 0$ since the right hand side has only positive terms. We can then set $z = k$ and plot the curves $4x^2 + y^2 = k$ on the xy-plane for various k. For $k = 0$ we see that the only x and y that satisfy $4x^2 + y^2 = 0$ are $x = y = 0$, so a single point. For other k we get ellipses as we see in Figure 2.24.

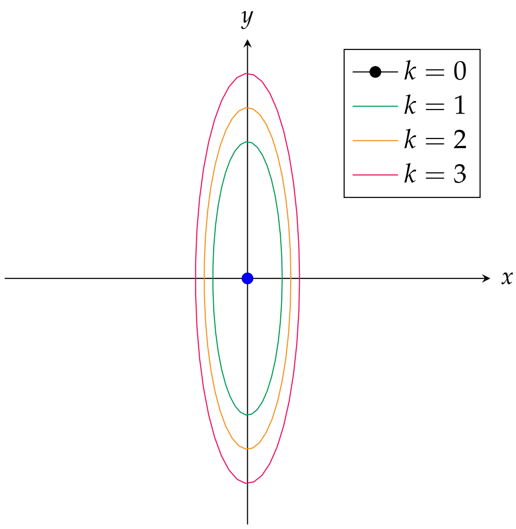


Figure 2.24: Level curves of the surface $z = 4x^2 + y^2$.

These are called **level curves** of the surface. In general we obtain them by writing a surface as $z = f(x, y)$ and then plotting the curve $f(x, y) = k$ on the xy-plan for various k. You can think of these as a heightmap (just like when reading a real map!), where different contours give the projection (to the xy-plane) of the curve of intersection of the surface $z = f(x, y)$ and the plane $z = k$. In any case, from the above plot of the level curves we can obtain the following graph of the surface.

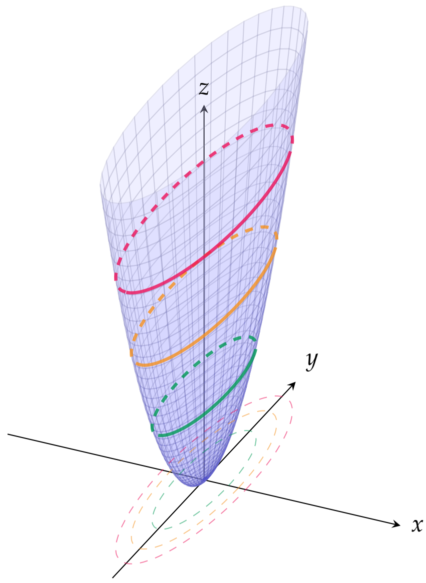


Figure 2.25: The elliptic paraboloid $z = 4x^2 + y^2$ along with some of the level curves lifted to the height of the plane of intersection, c.f. Figure 2.24.

This is called an *elliptic paraboloid*. Here we could try to predict the shape of the surface by observing that since the z variable is only linear then the outline of this surface should look like a parabola when viewed horizontally from any direction.

Example.

$$\boxed{z = x^2 - y^2}$$

This example is a bit trickier. Try for yourself to see if you can figure out what the surface should look like. Try plotting the level curves and also cut-offs along various axes. Eventually you should arrive at the following:

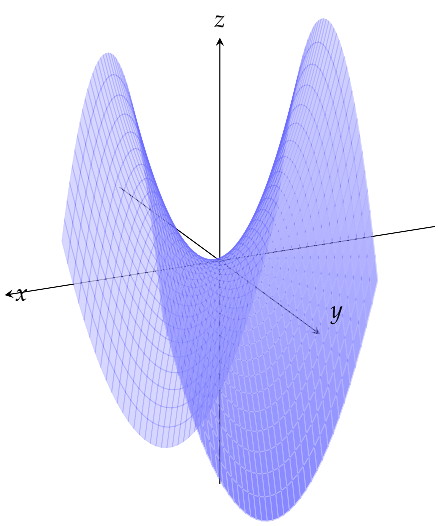


Figure 2.26: The hyperbolic paraboloid $z = x^2 - y^2$.

This surface looks like a saddle around the origin. It is called a *hyperbolic paraboloid*.

Example.

$$\boxed{z^2 = \frac{x^2}{2^2} + y^2}$$

When looking at the level curves of this surface notice that they are identical for $z = \pm k$. To get some more idea of what this surface looks like let's set $y = 0$, then $z^2 = x^2/2^2$ which gives, upon taking square roots, $z = \pm x/2$, which are just lines as in Figure 2.27.

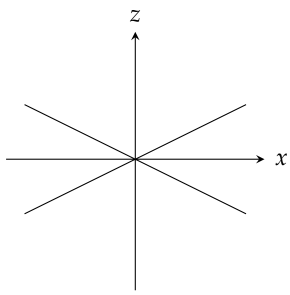


Figure 2.27: Vertical cut-off of the surface $z^2 = \frac{x^2}{2^2} + y^2$.

Now we should have the intuition that all the variables occur with the same exponents so in particular the z variable increases approximately linearly in x and y (as opposed to the paraboloid). With the help of this (and possibly some level curves that you should plot) we easily graph the surface.

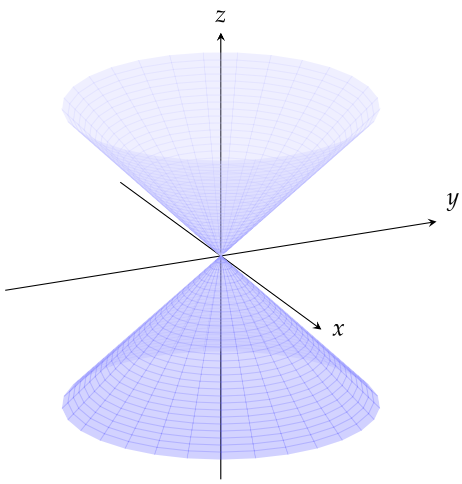


Figure 2.28: The equation $z^2 = \frac{x^2}{4} + y^2$ defines two cones.

Hence we see that we obtain a cone (or strictly speaking two cones).

Example.

$$\boxed{\begin{aligned} x^2 + y^2 - z^2 &= 1 \\ -x^2 - y^2 + z^2 &= 1 \end{aligned}}$$

For both of these surfaces notice that we are necessarily going to get some hyperbolas for our cross-sections because of the minus signs. Can you figure out what they are going to look like? It's useful to try and look at them "from the side" (fixing either x or y). These equations define what are called *one and two-sheeted hyperboloids*, respectively.

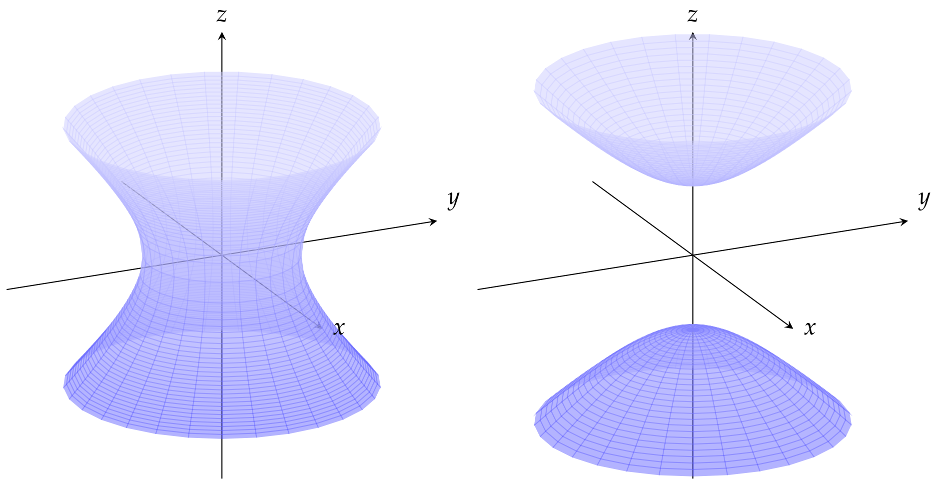


Figure 2.29: One and two-sheeted hyperboloids.

It is easy to remember which one is which by noticing that the number of minus signs in the defining equation matches with the number of sheets of the hyperboloid.

Chapter 3

Vector Valued Functions

In this chapter we will study functions of the form

$$\boldsymbol{r}\colon \mathbb{R} \to \mathbb{R}^3, \quad \boldsymbol{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}.$$

Here we think of t as a *parameter* (often e.g. time) and we call the scalar functions $x(t)$, $y(t)$ and $z(t)$ the *component functions* of $\boldsymbol{r}$. An example of such a function could be for example

$$\boldsymbol{r}(t) = \begin{pmatrix} t^2 \\ t^3 \\ \ln t \end{pmatrix}, \tag{3.1}$$

for $t > 0$. It is important to always specify the domain. If we don't specify the domain then it's typically assumed that it's the largest possible set where $\boldsymbol{r}(t)$ is defined. Also, from now on we will be pretty liberal with our notation for vectors by freely interchanging between the row and column notation.

This kind of functions can be used to represent, for example, the position of a particle in space or more generally to describe curves in three dimensions through *parametrisation*. The main focus in this chapter will be the use of this identification to compute properties of *smooth* curves in space.

3.1 Parametric Curves

Let $\boldsymbol{r}$ be as before. If $\boldsymbol{r}(t)$ is continuous then we can think of $\boldsymbol{r}(t)$ as tracing a curve in space by varying t. We give this as our first definition.

Definition 3.1. Let $\boldsymbol{r} : [a, b] \to \mathbb{R}^3$ (here a and b are possibly infinite) with $\boldsymbol{r}(t) = (x(t), y(t), z(t))$ be continuous. Then the image of the function $\boldsymbol{r}(t)$, $\{\boldsymbol{r}(t) : t \in [a, b]\}$, defines a curve in $\mathbb{R}^3$ denoted by $\mathscr{C}$. The function $\boldsymbol{r}(t)$ is called a **parametrisation** of $\mathscr{C}$.

When we say that $\boldsymbol{r}(t)$ is continuous we mean that it is continuous at every point $t_0 \in [a, b]$. This means that

$$\lim_{t\to t_0} \boldsymbol{r}(t) = \begin{pmatrix} \lim_{t\to t_0} x(t) \\ \lim_{t\to t_0} y(t) \\ \lim_{t\to t_0} z(t) \end{pmatrix} = \begin{pmatrix} x(t_0) \\ y(t_0) \\ z(t_0) \end{pmatrix} = \boldsymbol{r}(t_0),$$

so in other words we require that each of the component functions are continuous at every point in $[a, b]$. In general when taking limits, differentiating and integrating $\boldsymbol{r}(t)$ with respect to t, we can always do it component-wise.

Notice it's perfectly possible to have multiple parametrisations for the same curve. For example our curve in (3.1) can also be parametrised as

$$\boldsymbol{r}(u) = \begin{pmatrix} e^{2u} \\ e^{3u} \\ u \end{pmatrix}$$

where $u \in \mathbb{R}$. Therefore it makes sense to write just $\boldsymbol{r}$ for a curve $\mathscr{C}$ when talking about position along the curve (independent of parametrisation).

Example. Recall our example with $x = y = 3$. Then we already saw that this can be parametrised as (through the parametric equation of a line) as

$$\boldsymbol{r}(t) = \begin{pmatrix} 3 \\ 3 \\ t \end{pmatrix},$$

for $t \in \mathbb{R}$.

Here are some other curves that we saw in the class.

Example. This is a *trefoil knot* given by the parametrisation

$$\boldsymbol{r}(t) = \begin{pmatrix} (2 + \cos 1.5t) \cos t \\ (2 + \cos 1.5t) \sin t \\ \sin 1.5t \end{pmatrix}, \quad t \in [0, 4\pi].$$

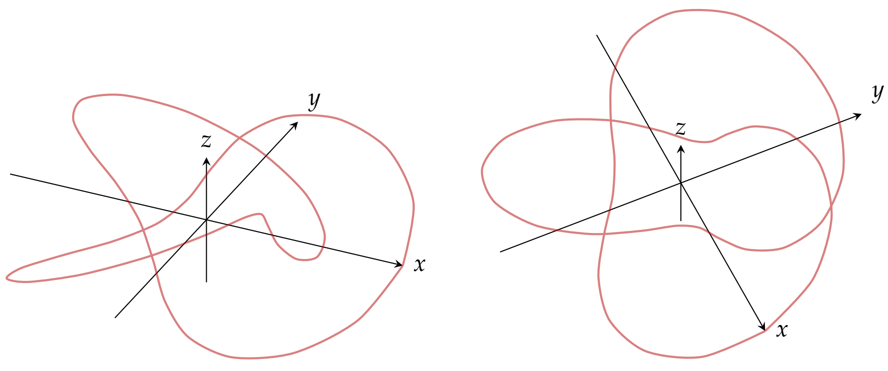


Figure 3.1: Two views of the trefoil knot.

Example. Here we have the *twisted cubic* given by

$$\boldsymbol{r}(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}, \quad -2 \leq t \leq 2.$$

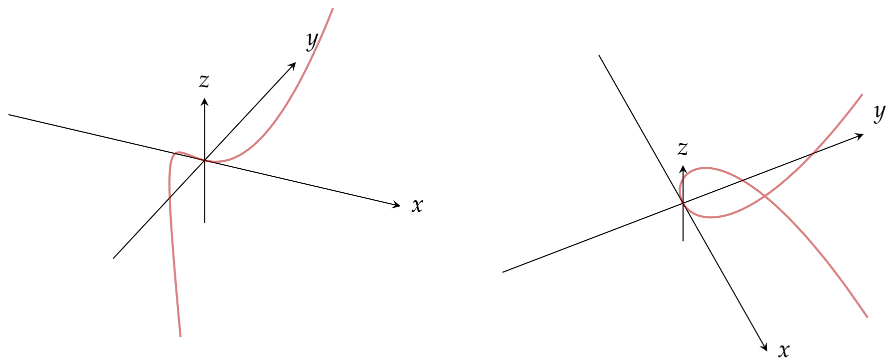


Figure 3.2: Two views of the twisted cubic.

A common situation where curves arise is as the intersection of surfaces.

Example. Find the curve of intersection of the cylinder $x^2 + 4y^2 = 4$ and the plane $x + 2y + 4z = 4$. The situation is as follows.

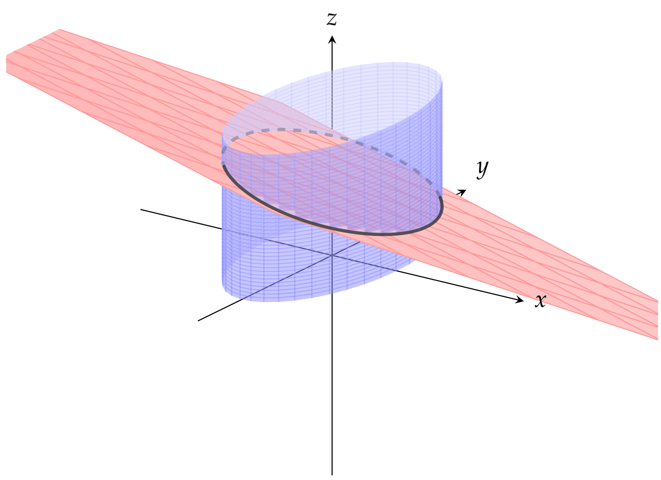


Figure 3.3: Intersection of a cylinder and a plane.

We have to find a parametrisation $\boldsymbol{r}(t) = (x(t), y(t), z(t))$ that satisfies the equations of both of the defining surfaces. We start by considering the equation of the cylinder as it doesn't depend on z, and then solve the second equation for z in terms of $x(t)$ and $y(t)$. A standard way to parametrise equations of the form $x^2 + 4y^2 = 4$ is to use the trigonometric identity $\sin^2 t + \cos^2 t = 1$, and to adjust the coefficients so we get what we want. Therefore, we set

$$\begin{aligned} x(t) &= 2\cos t, \\ y(t) &= \sin t. \end{aligned}$$

Generally we tend to drop the notation of dependence on t and just write x and y where

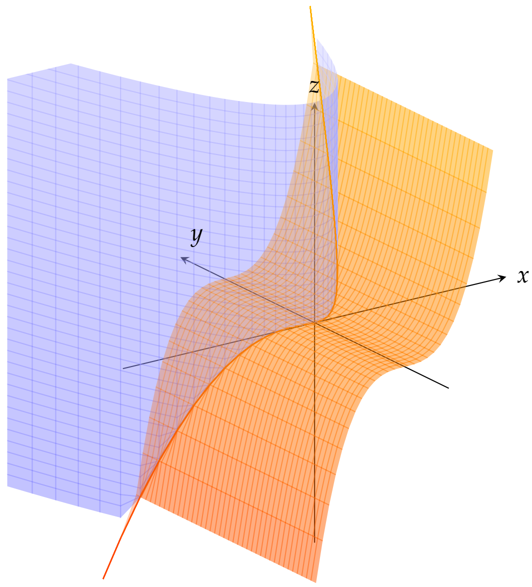


Figure 3.4: Twisted cubic as the intersection of the surfaces $y = x^2$ and $z = x^3$

it's understood they are functions of t. We then get from the equation of the plane that

$$z(t) = \frac{1}{4}(4 - x - 2y) = 1 - \frac{1}{2}(\cos t + \sin t).$$

Thus we obtain a parametrisation

$$\boldsymbol{r}(t) = \begin{pmatrix} 2\cos t \\ \sin t \\ 1 - \frac{1}{2}(\cos t + \sin t) \end{pmatrix}, \quad t \in [0, 2\pi).$$

Now show yourself that the twisted cubic curve is obtained as the curve of intersection of the surfaces $y = x^2$ and $z = x^3$, as seen in Figure 3.4.

3.2 Derivatives and Integrals of $\boldsymbol{r}(t)$

As we said before, it's easy to differentiate $\boldsymbol{r}(t)$, we just do it component-wise.

Definition 3.2. Let $\boldsymbol{r}(t) = (x(t), y(t), z(t))$. The derivative (if it exists) of $\boldsymbol{r}$ is

$$\boldsymbol{r}'(t) = \frac{d\boldsymbol{r}}{dt} = (x'(t), y'(t), z'(t)).$$

Similarly for integrating component-wise:

$$\int_a^b \boldsymbol{r}(t)dt = \begin{pmatrix} \int_a^b x(t)dt \\ \int_a^b y(t)dt \\ \int_a^b z(t)dt \end{pmatrix}.$$

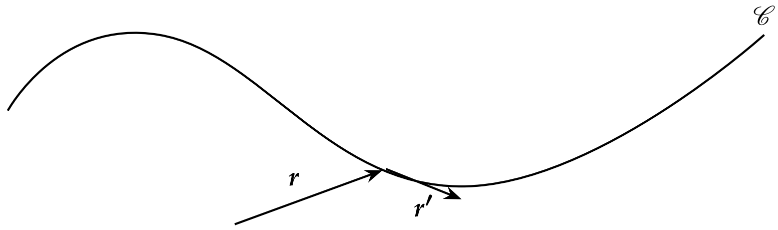


Figure 3.5: $\boldsymbol{r}'$ gives a tangent to the curve $\mathscr{C}$.

Example. If

$$\boldsymbol{r}(t) = (e^t, t^2, t),$$

then

$$\boldsymbol{r}'(t) = (e^t, 2t, 1).$$

In most of our examples our curves will always be *smooth*. For us this means that the derivative $\boldsymbol{r}'$ is *continuous* and *non-zero* ($\boldsymbol{r}' \neq \mathbf{0}$). This basically means that the curve has a well-defined direction at every point and that this direction changes continuously. Here are some basic properties of differentiation of vector-valued functions.

Theorem 3.3. *Let $\boldsymbol{r}(t)$, $\boldsymbol{s}(t)$ be vector-valued functions and let a, $b \in \mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$ be a scalar function. Then*

(i)

$$\frac{d}{dt}(a\boldsymbol{r}(t) + b\boldsymbol{s}(t)) = a\boldsymbol{r}'(t) + b\boldsymbol{s}'(t)$$

(ii)

$$\frac{d}{dt}(f(t)\boldsymbol{r}(t)) = f'(t)\boldsymbol{r}(t) + f(t)\boldsymbol{r}'(t)$$

(iii)

$$\frac{d}{dt}(\boldsymbol{r}(t) \bullet \boldsymbol{s}(t)) = \boldsymbol{r}'(t) \bullet \boldsymbol{s}(t) + \boldsymbol{r}(t) \bullet \boldsymbol{s}'(t)$$

(iv)

$$\frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{s}(t)) = \boldsymbol{r}'(t) \times \boldsymbol{s}(t) + \boldsymbol{r}(t) \times \boldsymbol{s}'(t)$$

(v)

$$\frac{d}{dt}(\boldsymbol{r}(f(t))) = f'(t)\boldsymbol{r}'(f(t)) \qquad \text{(Chain Rule)}$$

We note that physically $\boldsymbol{r}(t)$ gives the position on the curve $\mathscr{C}$, whereas $\boldsymbol{r}'(t)$ points in the direction of increasing t. This means that $\boldsymbol{r}'(t)$ is *tangent* to the curve $\mathscr{C}$.

Sometimes, if $\boldsymbol{r}$ describes the position of a particle, we call $\boldsymbol{r}'(t)$ *velocity* and denote it by $\boldsymbol{v}(t)$. Then $|\boldsymbol{r}'(t)| = v(t)$ is the *speed* of the particle. We will use these terms in the upcoming discussion to make matters more clear, but a more precise interpretation is postponed until the end of the chapter.

3.3 Arc Length and Curvature

Recommended problems from §13.3: **1, 17, 31**.

Now that we know how to describe curves, the first logical question is "*how long are they*". So let, as before, $\mathscr{C}$ be a curve parametrised by $\boldsymbol{r}(t) = (x(t), y(t), z(t))$ for $t \in [a, b]$. Without mentioning it further we now assume that all of our curves are *smooth*. To find the length of $\mathscr{C}$ from $\boldsymbol{r}(a)$ to $\boldsymbol{r}(b)$ we consider and arbitrary point along the curve, $\boldsymbol{r}(t)$ for some $t \in (a, b)$. Then, for small h, denote the length of the curve from $\boldsymbol{r}(t)$ to $\boldsymbol{r}(t+h)$ by ds. This is then roughly equal to the length of the line segment connecting $\boldsymbol{r}(t)$ and $\boldsymbol{r}(t+h)$ since h is small, as we see in Figure 3.6.

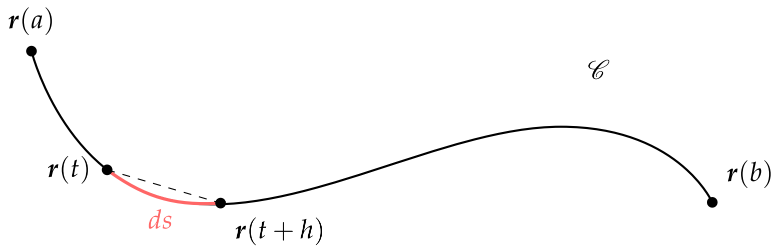


Figure 3.6: Approximating the length of the curve by line segments.

By a direct computation we then have

$$\begin{aligned} ds &\approx |\boldsymbol{r}(t+h) - \boldsymbol{r}(t)| \\ &= \sqrt{(x(t+h)-x(t))^2 + (y(t+h)-y(t))^2 + (z(t+h)-z(t))^2}. \end{aligned}$$

If we then let $h \to 0$ we obtain

$$ds = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt = |\boldsymbol{r}'(t)|dt. \tag{3.2}$$

This is not a completely rigorous argument, but for our purposes it should be sufficient to give an idea of what is going on and where the formulas come from. The quantity ds is called the **line element** of $\mathscr{C}$. In particular, we can rewrite (3.2) as

$$\boxed{\frac{ds}{dt} = |\boldsymbol{r}'(t)|.} \tag{3.3}$$

We can then compute the **arc length** of $\mathscr{C}$ by integrating ds along the curve.

Theorem 3.4. *Let $\mathscr{C}$ be given as before. Denote the length of $\mathscr{C}$ by s. Then*

$$s = \int_{\mathscr{C}} ds = \int_a^b |\boldsymbol{r}'(t)|dt = \int_a^b \left|\frac{d\boldsymbol{r}}{dt}\right| dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt. \tag{3.4}$$

This theorem demonstrates that our terminology agrees with the common physical idea that *total distance travelled is the integral of speed with respect to time*. Another important observation is that s does not depend on parametrisation, because it is expressed as the *distance* along the curve (take a moment to reflect on this and to convince yourself that this is true). We say that arc length is an *intrinsic* property of the curve.

Example. Find the length of the curve

$$\boldsymbol{r}(t) = (t^2, t^2, t^3)$$

from $t = 0$ to $t = 1$. To apply (3.4) we compute the derivative (velocity) as

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{r}'(t) = (2t, 2t, 3t^3).$$

Taking the norm (length) of this gives

$$|\boldsymbol{r}'(t)| = \sqrt{4t^2 + 4t^2 + 9t^4} = \sqrt{8t^2 + 9t^4} = t\sqrt{8 + 9t^2}.$$

Then

$$\begin{aligned} s &= \int_0^1 \left|\frac{d\boldsymbol{r}}{dt}\right| dt \\ &= \int_0^1 t\sqrt{8 + 9t^2}\, dt && (\text{let } 8 + 9t^2 = u) \\ &= \frac{1}{18}\int_8^{17} \sqrt{u}du \\ &= \frac{1}{18}\left[\frac{2u^{3/2}}{3}\right]_8^{17}. \end{aligned}$$

So after simplifying we arrive at

$$s = \frac{1}{27}(17\sqrt{17} - 16\sqrt{2})$$

as the length of the curve from 0 to 1.

We emphasise again that *any* parametrisation of the curve gives the same result. Can you come up with another parametrisation of the above curve? Compute the length of the curve by using that parametrisation. We'll next introduce an example that will be recurring throughout the rest of this chapter.

Example. Let $\mathscr{C}$ be parametrised as

$$\boldsymbol{r}(t) = (a\cos t, a\sin t, bt),$$

for some constants $a, b \in \mathbb{R}$. This is called a *circular helix* (circular because both the cos and the sin have the same coefficient, otherwise we just say *helix*). What is the length of $\mathscr{C}$ from $t = 0$ to $t = 2\pi$? We start by computing the line element:

$$ds = \left|\frac{d\boldsymbol{r}}{dt}\right| dt = |(-a\sin t, a\cos t, b)| dt = \sqrt{a^2 + b^2}\, dt.$$

This is particularly handy since ds is independent of t! This is why the helix is a very nice example. It is then easy to see that the length of the curve is

$$s = \int_0^{2\pi} \left|\frac{d\boldsymbol{r}}{dt}\right| dt = \int_0^{2\pi} \sqrt{a^2 + b^2}\, dt = 2\pi\sqrt{a^2 + b^2}.$$

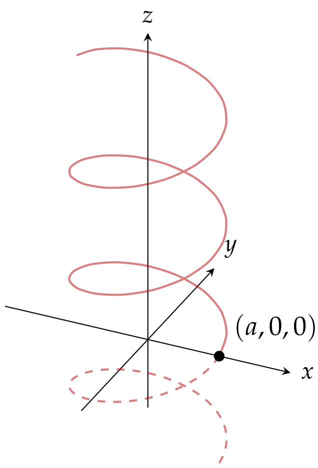


Figure 3.7: A circular helix.

In general it can be quite tricky to compute the integral $\int ds$ for the length of the curve. There is another use for our introducing the arc length though. Recall how it is an intrinsic property of the curve. Thus it provides a somehow "*natural*" (but generally difficult) way to parametrise a curve. The main idea is to fix a point along the curve and then use the (signed) distance from that point measured along the curve (i.e. the arc length s) as the paramter. So more precisely we arrive at $\boldsymbol{r}(s)$ which gives the point on the curve $\mathscr{C}$ at arc length s away from $\boldsymbol{r}(0)$. Then $\boldsymbol{r}(s)$ is called the **arc length parametrisation** (or **intrinsic parametrisation**) of $\mathscr{C}$. As we pointed out before, this parametrisation is independent of the coordinate system and is well-defined for all smooth curves. Now, recall from (3.3) that

$$ds = |\boldsymbol{r}'(t)|dt$$

for any parametrisation $\boldsymbol{r}(t)$ of the curve $\mathscr{C}$. Thus for the arc length parametrisation we obtain

$$ds = |\boldsymbol{r}'(s)|ds, \tag{3.5}$$

which means that $|\boldsymbol{r}'(s)| = 1$ so that with the arc length parametrisation $\mathscr{C}$ is *travelled at unit speed*. Like with the length of a curve, the arc length parametrisation is also difficult to find in general. The general strategy is to take a curve $\mathscr{C}$ with some parametrisation $\boldsymbol{r}(t)$. We then write the distance as a function of t as

$$s = s(t) = \int_a^t \left|\frac{d\boldsymbol{r}(u)}{du}\right| du,$$

from some point a. If we can then solve this for t in terms of s we can then substitute the answer back to $\boldsymbol{r}(t(s)) = \boldsymbol{r}(s)$ to obtain a parametrisation in terms of s. An example will make this more clear. Luckily for our favourite example of the helix, it's straightforward.

Example. Let again

$$\boldsymbol{r}(t) = (a\cos t, a\sin t, bt).$$

Recall we computed the line element as

$$ds = \sqrt{a^2 + b^2}\,dt.$$

Thus

$$s(t) = \int_0^t \sqrt{a^2+b^2}du = \sqrt{a^2+b^2}t,$$

and it follows that

$$t = \frac{s}{\sqrt{a^2+b^2}}.$$

Substituting back to $\boldsymbol{r}(t)$ we get

$$\boldsymbol{r}(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}}\right).$$

In summary, $\boldsymbol{r}(s)$ gives the point on $\mathscr{C}$ at distance s (measured along $\mathscr{C}$) from $\boldsymbol{r}(0) = (a, 0, 0)$.

We'll now see the power of the arc length parametrisation by deducing some useful concepts for general curves. In particular we want to define a coordinate system for $\mathbb{R}^3$ at any point on $\mathscr{C}$ that only depends on s. It follows immediately that all the following properties are independent of the parametrisation of the curve.

To begin with recall that $\boldsymbol{r}'(t)$ is a tangent vector to the curve $\mathscr{C}$ (as long as $\boldsymbol{r}'(t) \neq \boldsymbol{0}$) for any parametrisation. But then we showed in (3.5) that for the arc length parametrisation this vector is of unit length. We call this the **unit tangent vector** to $\mathscr{C}$ and denote it by $\hat{\boldsymbol{T}}$, i.e.

$$\hat{\boldsymbol{T}} = \frac{d\boldsymbol{r}}{ds}.$$

In terms of an arbitrary parametrisation this is then

$$\boxed{\hat{\boldsymbol{T}} = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|},}$$

as long as $\boldsymbol{r}'(t) \neq \boldsymbol{0}$ (which it always is for smooth curves). Now, since $|\hat{\boldsymbol{T}}| = 1 = \hat{\boldsymbol{T}} \cdot \hat{\boldsymbol{T}}$ we can differentiate this w.r.t s to get

$$\frac{d\hat{\boldsymbol{T}}}{ds} \cdot \hat{\boldsymbol{T}} + \hat{\boldsymbol{T}} \cdot \frac{d\hat{\boldsymbol{T}}}{ds} = 0. \tag{3.6}$$

It follows that the vector $\frac{d\hat{\boldsymbol{T}}}{ds}$ is perpendicular to the unit tangent $\hat{\boldsymbol{T}}$. The length of this vector has a special meaning.

Definition 3.5. The **curvature** κ of $\mathscr{C}$ is defined as

$$\kappa = \kappa(s) = \left|\frac{d\hat{\boldsymbol{T}}}{ds}\right|.$$

The curvature measures roughly speaking the rate at which $\mathscr{C}$ is "turning" (away from the tangent line) at any point. Notice that since $|\hat{\boldsymbol{T}}| = 1$ it follows that any change to κ occurs strictly from the change in the direction of $\hat{\boldsymbol{T}}$. By the above definition and the observation that $\frac{d\hat{\boldsymbol{T}}}{ds} \perp \hat{\boldsymbol{T}}$ we set

$$\boxed{\frac{d\hat{\boldsymbol{T}}}{ds} = \kappa\hat{\boldsymbol{N}},} \tag{3.7}$$

as long as $\kappa \neq 0$. In other words $\hat{\boldsymbol{N}} = \frac{d\hat{\boldsymbol{T}}}{ds} / |\frac{d\hat{\boldsymbol{T}}}{ds}|$ is the unit vector in the direction of $\frac{d\hat{\boldsymbol{T}}}{ds}$. The vector $\hat{\boldsymbol{N}}$ is called the **unit normal vector** to $\mathscr{C}$ and it is perpendicular to $\hat{\boldsymbol{T}}$. It is only defined if $\kappa \neq 0$. For example for a straight line the curvature is 0 so it doesn't have a unit normal vector. We now have two unit vectors defined in terms of the arc length of $\mathscr{C}$. In order to have a coordinate system we need a third vector. A natural choice is

$$\hat{\boldsymbol{B}} = \hat{\boldsymbol{T}} \times \hat{\boldsymbol{N}}, \tag{3.8}$$

which is called the **unit binormal vector**. Together the vectors $\hat{\boldsymbol{T}}$, $\hat{\boldsymbol{N}}$ and $\hat{\boldsymbol{B}}$ form what we call the **Frenet frame** (or just the TNB-frame), $(\hat{\boldsymbol{T}}, \hat{\boldsymbol{N}}, \hat{\boldsymbol{B}})$, which is a basis for $\mathbb{R}^3$. It is useful to notice that in the definition of $\hat{\boldsymbol{B}}$ we can cycle the letters left or right (without changing the order, i.e. even permutations)

$$\boxed{\hat{\boldsymbol{B}} \times \hat{\boldsymbol{T}} = \hat{\boldsymbol{N}}, \qquad \hat{\boldsymbol{N}} \times \hat{\boldsymbol{B}} = \hat{\boldsymbol{T}}.} \tag{3.9}$$

By the same argument as in (3.6) we see that

$$\frac{d\hat{\boldsymbol{B}}}{ds} \perp \hat{\boldsymbol{B}},$$

and thus if we differentiate $\hat{\boldsymbol{B}} = \hat{\boldsymbol{T}} \times \hat{\boldsymbol{N}}$ (w.r.t. s) we get

$$\begin{aligned}
\frac{d\hat{\boldsymbol{B}}}{ds} &= \frac{d\hat{\boldsymbol{T}}}{ds} \times \hat{\boldsymbol{N}} + \hat{\boldsymbol{T}} \times \frac{d\hat{\boldsymbol{N}}}{ds} \\
&= \kappa\hat{\boldsymbol{N}} \times \hat{\boldsymbol{N}} + \hat{\boldsymbol{T}} \times \frac{d\hat{\boldsymbol{N}}}{ds} && \text{(by (3.7))} \\
&= \hat{\boldsymbol{T}} \times \frac{d\hat{\boldsymbol{N}}}{ds}.
\end{aligned}$$

We deduce that $\frac{d\hat{\boldsymbol{B}}}{ds}$ is perpendicular to $\hat{\boldsymbol{T}}$. Since we know it's also perpendicular to $\hat{\boldsymbol{B}}$, it follows necessarily that $\frac{d\hat{\boldsymbol{B}}}{ds}$ is parallel to $\hat{\boldsymbol{N}}$. That is $\frac{d\hat{\boldsymbol{B}}}{ds}$ is a scalar multiple of $\hat{\boldsymbol{N}}$. The scalar in question has a special meaning.

Definition 3.6. The **torsion** τ of a curve $\mathscr{C}$ is given by

$$\frac{d\hat{\boldsymbol{B}}}{ds} = -\tau(s)\hat{\boldsymbol{N}} = -\tau\hat{\boldsymbol{N}}, \tag{3.10}$$

which is well-defined for any smooth curve as long as $\kappa(s) \neq 0$.

The torsion measures how much the curve "twists", that is how much it fails to be planar (contained in a single plane) at s.

Since the arc length parametrisation is generally difficult to find, the above formulas are not that useful to work with. We'll next write them in terms of a general parametrisation $\boldsymbol{r} = \boldsymbol{r}(t)$. We already showed this for the unit tangent vector. We can be more explicit about it by using the relation $ds = |\frac{d\boldsymbol{r}}{dt}|dt$, thus

$$\hat{\boldsymbol{T}} = \frac{d\boldsymbol{r}}{ds} = \frac{\frac{d\boldsymbol{r}}{dt}}{\left|\frac{d\boldsymbol{r}}{dt}\right|} = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|},$$

by the Chain Rule. Then for the curvature we get

$$\kappa = \left|\frac{d\hat{\boldsymbol{T}}}{ds}\right| = \left|\frac{d\hat{\boldsymbol{T}}}{dt}\frac{dt}{ds}\right| = \left|\frac{\hat{\boldsymbol{T}}'(t)}{\boldsymbol{r}'(t)}\right|,$$

again by the Chain Rule. This is still not quite there, we want to get a formula only in terms of $\boldsymbol{r}(t)$. So let's write $\boldsymbol{r}'$ in terms of t:

$$\boldsymbol{r}' = |\boldsymbol{r}'(t)|\hat{\boldsymbol{T}} = \frac{ds}{dt}\hat{\boldsymbol{T}}.$$

Differentiating w.r.t. t gives:

$$\boldsymbol{r}'' = \frac{d^2s}{dt^2}\hat{\boldsymbol{T}} + \frac{ds}{dt}\frac{d\hat{\boldsymbol{T}}}{dt}.$$

But we know $\frac{d\hat{\boldsymbol{T}}}{dt} = \frac{d\hat{\boldsymbol{T}}}{ds}\frac{ds}{dt}$ (Chain Rule) so

$$\begin{aligned} \boldsymbol{r}'' &= \frac{d^2 s}{dt^2}\hat{\boldsymbol{T}} + \left(\frac{ds}{dt}\right)^2 \frac{d\hat{\boldsymbol{T}}}{ds} \\ &= \frac{d^2 s}{dt^2}\hat{\boldsymbol{T}} + \left(\frac{ds}{dt}\right)^2 \kappa\hat{\boldsymbol{N}}. \end{aligned}$$

By taking the cross product we arrive at (by orthogonality)

$$\boldsymbol{r}' \times \boldsymbol{r}'' = \left(\frac{ds}{dt}\right)^3 \kappa\hat{\boldsymbol{T}} \times \hat{\boldsymbol{N}} = \left(\frac{ds}{dt}\right)^3 \kappa\hat{\boldsymbol{B}}. \tag{3.11}$$

From this we can deduce the following two formulae. First by comparing the magnitudes of the vectors on each side of equation (3.11), we get a formula for the curvature.

Theorem 3.7.

$$\kappa = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{|\boldsymbol{r}'|^3}. \tag{3.12}$$

Then, the vector on the right-hand side of (3.11) of course points in the direction of $\hat{\boldsymbol{B}}$, so therefore the vector on the left-hand side has to as well. Moreover, $\hat{\boldsymbol{B}}$ is a unit vector so we get a formula for $\hat{\boldsymbol{B}}$ by normalising the left-hand side.

Theorem 3.8.

$$\hat{\boldsymbol{B}} = \frac{\boldsymbol{r}' \times \boldsymbol{r}''}{|\boldsymbol{r}' \times \boldsymbol{r}''|}. \tag{3.13}$$

It is also possible to find a general formula for the torsion. We won't derive it here, if you are curious have a look at Exercise 63 in section 13.4 in the book.

Theorem 3.9.

$$\tau = \frac{(\boldsymbol{r}' \times \boldsymbol{r}'') \bullet \boldsymbol{r}'''}{|\boldsymbol{r}' \times \boldsymbol{r}''|^2}. \tag{3.14}$$

Let's do an example of finding the Frenet frame. Since we have formulas for $\hat{\boldsymbol{T}}$ and $\hat{\boldsymbol{B}}$ in terms of (derivatives) of $\boldsymbol{r}$, it's usually easiest to compute these first and then use $\hat{\boldsymbol{N}} = \hat{\boldsymbol{B}} \times \hat{\boldsymbol{T}}$ to find the unit normal vector. Alternatively you could also try to find the unit normal from $\hat{\boldsymbol{N}} = \frac{\hat{\boldsymbol{T}}'}{|\hat{\boldsymbol{T}}'|}$ and then find the unit bi-normal through the vector product. We use our favourite example, the helix.

Example. Let, as before,

$$\boldsymbol{r}(t) = (a\cos t, a\sin t, bt),$$

where $t \in \mathbb{R}$ and $a > 0$. To compute $\hat{\boldsymbol{T}}$ we need $\boldsymbol{r}'$, so

$$\begin{aligned} \boldsymbol{r}'(t) &= (-a\sin t, a\cos t, b), \\ |\boldsymbol{r}'(t)| &= \sqrt{a^2 + b^2}. \end{aligned}$$

Thus

$$\hat{\boldsymbol{T}} = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|} = \left(\frac{-a\sin t}{\sqrt{a^2+b^2}}, \frac{a\cos t}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right).$$

Next, we find $\hat{\boldsymbol{B}}$ through (3.13), so we need to compute $\boldsymbol{r}''$.

$$\begin{aligned}
\boldsymbol{r}''(t) &= (-a\cos t, -a\sin t, 0),\\
\boldsymbol{r}' \times \boldsymbol{r}'' &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = \begin{pmatrix} ab\sin t \\ -ab\cos t \\ a^2 \end{pmatrix},\\
|\boldsymbol{r}' \times \boldsymbol{r}''| &= \sqrt{(ab)^2(\sin^2 t + \cos^2 t) + a^4} = \sqrt{a^2(a^2+b^2} = a\sqrt{a^2+b^2},
\end{aligned}$$

since $a > 0$. Therefore

$$\hat{\boldsymbol{B}} = \left(\frac{b\sin t}{\sqrt{a^2+b^2}}, \frac{-b\cos t}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}\right).$$

Then we compute the cross product to find $\hat{\boldsymbol{N}}$,

$$\hat{\boldsymbol{N}} = \hat{\boldsymbol{B}} \times \hat{\boldsymbol{T}} = \frac{1}{a^2+b^2}\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ b\sin t & -b\cos t & a \\ -a\sin t & a\cos t & b \end{vmatrix} = \frac{1}{a^2+b^2}\begin{pmatrix} -(a^2+b^2)\cos t \\ -(a^2+b^2)\sin t \\ 0 \end{pmatrix}.$$

We simplify to get

$$\hat{\boldsymbol{N}} = (-\cos t, -\sin t, 0).$$

Notice that the unit normal vector is independent of a and b! It points towards the centre of helix, which is the z-axis. Check yourself that you get the same $\hat{\boldsymbol{N}}$ if you differentiate $\hat{\boldsymbol{T}}$ and normalise. To find the curvature we use (3.12). Notice that we have already more or less computed the necessary quantities, so this is fast.

$$\kappa = \kappa(t) = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{|\boldsymbol{r}'|^3} = \frac{a\sqrt{a^2+b^2}}{(a^2+b^2)^{3/2}} = \frac{a}{a^2+b^2}.$$

As we pointed out before, you could also use the intrinsic parametrisation of the helix to find the $\boldsymbol{TNB}$-frame and the curvature. You should do this. Finally, for planar curves (curves that are completely contained in some fixed plane) the curvature formula has a particularly nice form. Let's suppose that $\mathscr{C}$ is contained in the xy-plane and is given by the equation $y = f(x)$ (so $z = 0$). Then we can parametrise $\mathscr{C}$ as

$$\boldsymbol{r}(t) = (x, f(x), 0),$$

for $x \in \mathbb{R}$. Let's compute the curvature through (3.12).

$$\begin{aligned}
\boldsymbol{r}'(x) &= (1, f'(x), 0), & |\boldsymbol{r}'(x)| &= \sqrt{1 + f'(x)^2},\\
\boldsymbol{r}''(x) &= (0, f''(x), 0), & \boldsymbol{r}' \times \boldsymbol{r}''(x) &= (0, 0, f''(x)).
\end{aligned}$$

Putting everything together gives

$$\boxed{\kappa = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}}.} \tag{3.15}$$

Example. Find curvature of $y = x^2$ at (x, x^2).

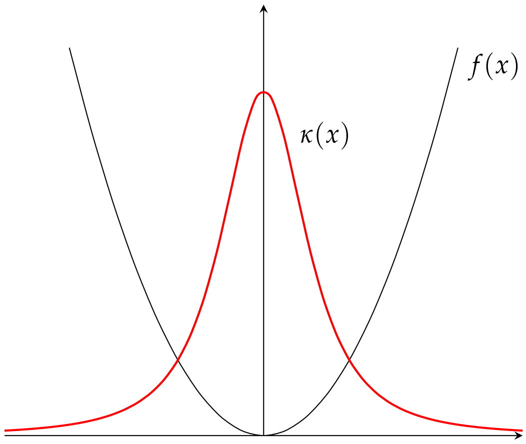


Figure 3.8: The curve $y = f(x) = x^2$ and its curvature in red.

So in this example we have $f(x) = x^2$, and thus $f'(x) = 2x$ and $f''(x) = 2$. Thus by (3.15)

$$\kappa(x) = \frac{2}{(1 + 4x^2)^{3/2}}.$$

We can see the curve $f(x)$ and the curvature plotted in Figure 3.8. Notice that the curvature is highest at $x = 0$, that is when the curve is "turning" the most. On the other hand, as $x \to \pm\infty$, the curvature vanishes since the curve $y = f(x)$ becomes more and more flat (vertical).

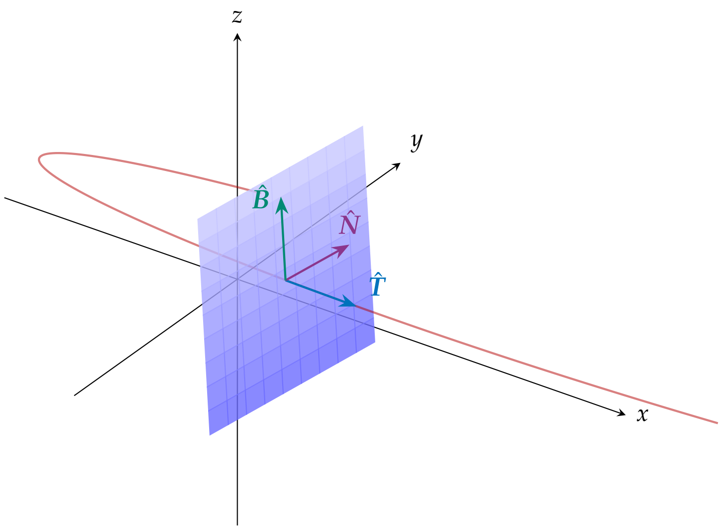


Figure 3.9: Normal plane to the curve $\mathscr{C}$ at P is parallel to $\hat{\boldsymbol{N}}$ and $\hat{\boldsymbol{B}}$.

In mathematics it is often helpful to be able to reduce problems to a lower dimension. Consider for example the derivative of a function of one variable. The graph of the function is a one dimensional object (as it depends on only one variable), but it is embedded naturally only in two dimensions. We then approximate the curve near a fixed point through a linearisation given by the derivative. This simplifies the curve into a line, which comfortably sits inside one dimension only.

We now want to do something similar for curves inside 3-space. Let $\mathscr{C}$ be a curve and fix a point P on the curve. The plane orthogonal to $\hat{\boldsymbol{T}}$ at P is called the **normal plane** of $\mathscr{C}$, as shown in Figure 3.9. If $\mathscr{C}$ is parametrised by $\boldsymbol{r}(t)$, then this is equivalent to saying that the normal plane is perpendicular to $\boldsymbol{r}'(t)$ (why?). It follows from the vector equation of a plane that the equation for the normal plane can be written as

$$\boldsymbol{r}'(t) \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \boldsymbol{r}(t) \right) = 0, \tag{3.16}$$

where $\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)$ is an arbitrary point on the normal plane and thus $\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right) - \boldsymbol{r}(t)$ is a vector that lies on the plane. Notice that if the curvature κ is non-zero then $\mathscr{C}$ has well-defined unit normal and bi-normal vectors. The normal plane is then the plane defined by $\hat{\boldsymbol{N}}$ and $\hat{\boldsymbol{B}}$. It should be evident that at any point P, the curve passes perpendicularly through the normal plane.

Let's suppose for now that $\kappa \neq 0$, then $\mathscr{C}$ has a well-defined unit bi-normal. The plane orthogonal to $\hat{\boldsymbol{B}}$ at P is called the **osculating plane**, see Figure 3.10. Equivalently it is the plane defined by $\hat{\boldsymbol{T}}$ and $\hat{\boldsymbol{N}}$. To find an equation for this plane we note that $\boldsymbol{r}' \times \boldsymbol{r}''$ is parallel to $\hat{\boldsymbol{B}}$. Thus the osculating plane is given by

$$\left(\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\right) \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \boldsymbol{r}(t) \right) = 0. \tag{3.17}$$

This is the plane that contains the most of $\mathscr{C}$ near P, so if we look in really close near

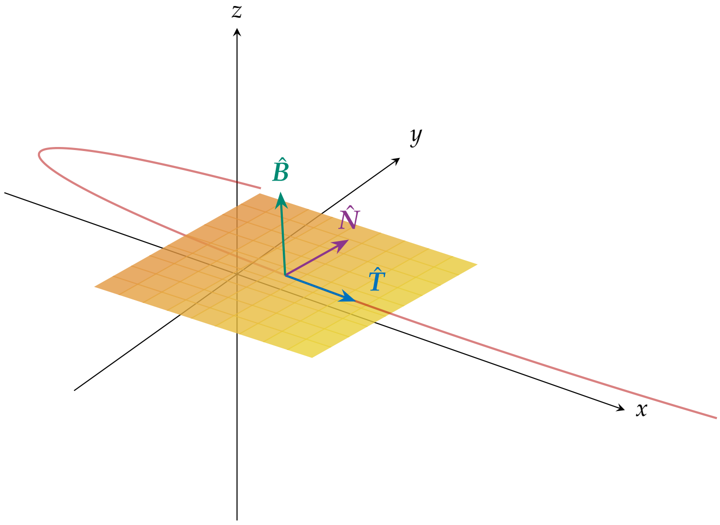


Figure 3.10: The osculating plane to $\mathscr{C}$ at P is parallel to $\hat{\boldsymbol{T}}$ and $\hat{\mathbf{N}}$.

the point P then $\mathscr{C}$ lies on the osculating plane. In this sense we can say that locally the curve resembles a planar curve. In particular, for any planar curve in three dimensions, the plane containing the curve *is* the osculating plane.

Now we'll restrict to working on the osculating plane (so this discussion applies to curves in two dimensions by analogy). Then, draw a circle on the *concave* side of $\mathscr{C}$ (this is well-defined since $\kappa \neq 0$) which passes through P and two points infinitesimally close to P on $\mathscr{C}$. As the distance of these two points from P tends to zero, we obtain what is called the **osculating circle**. It is a circle that intersects $\mathscr{C}$ at P and has the same tangent at that point. By definition $\hat{\mathbf{N}}$ points towards the centre of this circle. It's also possible to show that the radius of the osculating circle is the inverse of the curvature, κ^{-1}. We give this quantity a special name.

Definition 3.10. The **radius of curvature** of a curve $\mathscr{C}$ is

$$\rho = \rho(s) = \frac{1}{\kappa(s)},$$

if $\kappa(s) \neq 0$.

Here we've just highlighted the fact that the curvature (and radius of curvature) change when you move along the curve (by depending on s). The centre of the osculating circle is given by the vector

$$\boldsymbol{r} + \frac{1}{\kappa}\hat{\mathbf{N}}.$$

Moreover, the osculating circle and $\mathscr{C}$ have the same curvature at P. Therefore the osculating circle is a straight extension of the idea of a tangent line approximation to a curve. If, for the moment, we think of a plane curve given by the equation $y = f(x)$, then the osculating circle is essentially a "second degree" approximation to the graph of f at some point $x = a$. That is, we are saying that the first as well as the *second* derivatives of f agree with those of the osculating circle at $x = a$. First derivatives matching ensures equal tangent lines, while the equal second derivatives force the curvatures to match.

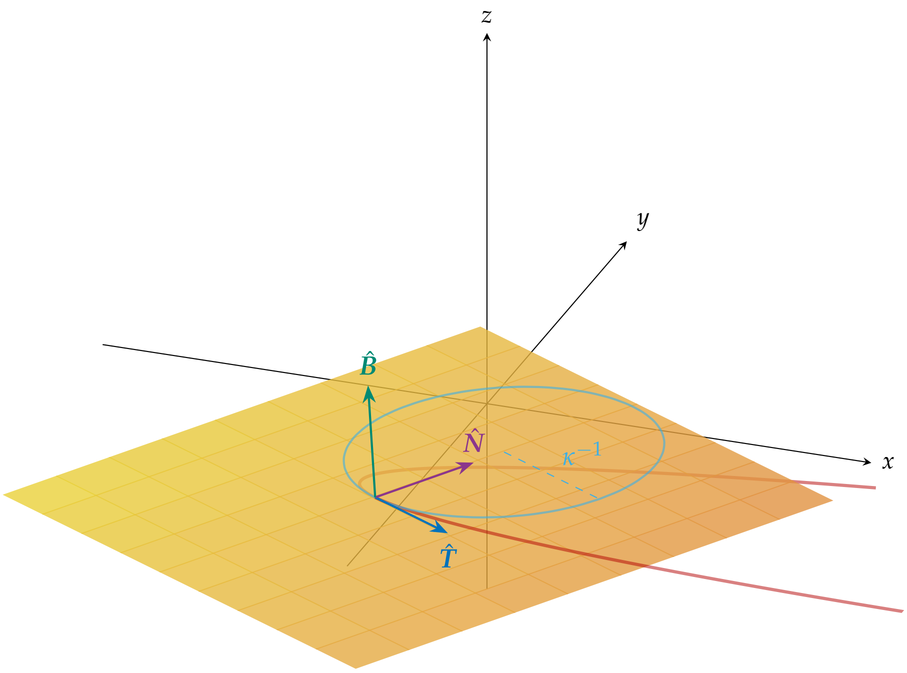


Figure 3.11: The osculating circle approximates $\mathscr{C}$ near P.

While the tangent is the "best" line that approximates the curve, the osculating circle is the "best" circle that approximates the curve.

Example. Find equations of the normal plane, osculating plane and the osculating circle for the helix:

$$\boldsymbol{r}(t) = (a\cos t, a\sin t, bt),$$

where $a > 0$. This is a straightforward computation. As before,

$$\boldsymbol{r}'(t) = (-a\sin t, a\cos t, b).$$

Thus by (3.16) the normal plane is given by

$$\begin{pmatrix} -a\sin t \\ a\cos t \\ b \end{pmatrix} \cdot \begin{pmatrix} x - a\cos t \\ y - a\sin t \\ z - bt \end{pmatrix} = 0,$$

which rearranges to

$$(-a\sin t)x + (a\cos t)y + bz = b^2 t.$$

For the osculating plane we compute

$$\boldsymbol{r}''(t) = (-a\cos t, -a\sin t, 0),$$
$$\boldsymbol{r}' \times \boldsymbol{r}'' = (ab\sin t, ab\cos t, a^2),$$

and hence by (3.17) the equation for the osculating plane is

$$(b\sin t)x - (b\cos t)y + az = abt,$$

since $a > 0$. For the osculating circle we need to compute the unit normal vector, to do this we can, for example, find $\hat{\boldsymbol{T}}$ and $\hat{\boldsymbol{B}}$ first

$$\hat{\boldsymbol{T}} = \frac{\boldsymbol{r}'}{|\boldsymbol{r}'|} = \frac{1}{\sqrt{a^2+b^2}}(-a\sin t, a\cos t, b),$$
$$\hat{\boldsymbol{B}} = \frac{\boldsymbol{r}' \times \boldsymbol{r}''}{|\boldsymbol{r}' \times \boldsymbol{r}''|} = \frac{1}{a\sqrt{a^2+b^2}}(ab\sin t, -ab\cos t, a^2),$$

and then compute $\hat{\boldsymbol{N}}$ by taking the cross product:

$$\hat{\boldsymbol{N}} = \hat{\boldsymbol{B}} \times \hat{\boldsymbol{T}} = (-\cos t, -\sin t, 0).$$

We also need the curvature:

$$\kappa = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{|\boldsymbol{r}'|^3} = \frac{a\sqrt{a^2+b^2}}{(a^2+b^2)^{3/2}} = \frac{a}{a^2+b^2}.$$

Thus the osculating circle has radius $\rho = \kappa^{-1} = a^{-1}(a^2+b^2)$ and centre

$$\boldsymbol{r}(t) + \frac{1}{\kappa}\hat{\boldsymbol{N}} = (a\cos t, a\sin t, bt) + \frac{a^2+b^2}{a}(-\cos t, -\sin t, 0)$$
$$= (-\frac{b^2}{a}\cos t, -\frac{b^2}{a}\sin t, bt).$$

Notice that if $b = 0$ then $\boldsymbol{r}$ traces out a circle C of radius a on the xy-plane. In this case we see that the osculating circle is just the circle C itself and the vector for the centre becomes $\boldsymbol{0}$ as you would expect. In Figure 3.12 you can see the normal plane (in blue), osculating plane (in yellow) and the osculating circle on the osculating plane (in cyan) for the helix.

3.4 Components of Acceleration

Recommended problems from §13.4: **7, 37, 39**.

Let's now go back to the idea that we can think of $\boldsymbol{r}(t)$ as giving the position of a particle in space, where we consider the parameter t as *time*. We already mentioned before that in this case the derivative of $\boldsymbol{r}$ gives the velocity, we write

$$\boldsymbol{r}'(t) = \boldsymbol{v}(t).$$

And thus the *speed* of the particle is $|\boldsymbol{v}(t)|$. It follows that the acceleration of the particle is

$$\boldsymbol{r}''(t) = \boldsymbol{a}(t).$$

We know that $\boldsymbol{r}'$ points in the direction of the unit tangent vector $\hat{\boldsymbol{T}}$. But what about $\boldsymbol{a}$? A common mistake is to think that $\boldsymbol{r}''$ has to point in the direction of the normal vector,

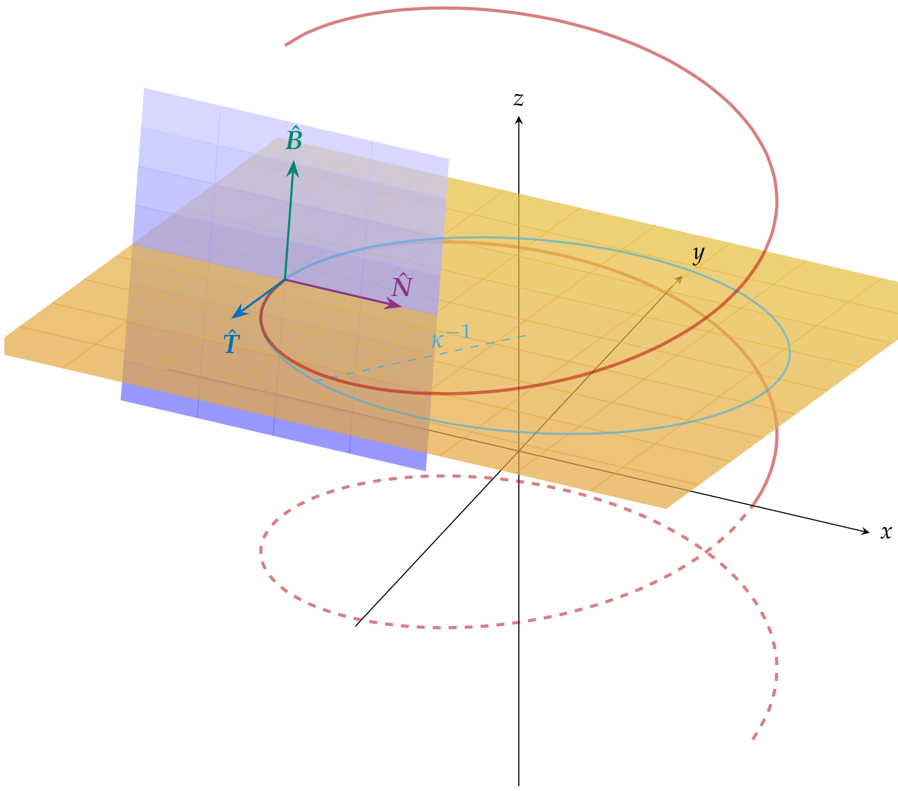


Figure 3.12: Normal plane, osculating plane and the osculating circle for the helix.

but this is not typically the case. In fact, we've already done all the computations to see what is actually going on. Recall that in the last lecture we proved that

$$\boldsymbol{r}'' = \frac{d^2 s}{dt^2}\hat{\boldsymbol{T}} + \left(\frac{ds}{dt}\right)^2 \kappa \hat{\boldsymbol{N}}.$$

In our new notation this reads

$$\boldsymbol{a} = \frac{d}{dt}|\boldsymbol{v}|\hat{\boldsymbol{T}} + |\boldsymbol{v}|^2 \kappa \hat{\boldsymbol{N}}, \tag{3.18}$$

or just

$$\text{acceleration} = \frac{d}{dt}(\text{speed})\hat{\boldsymbol{T}} + (\text{speed})^2 \kappa \hat{\boldsymbol{N}}.$$

From this we see that the acceleration vector only has components in the tangential and normal directions, with no component in the $\hat{\boldsymbol{B}}$ direction. Denote the *magnitudes* of the tangential and normal components of $\boldsymbol{a}$ by a_T and a_N, respectively, that is

$$\boldsymbol{r}'' = \boldsymbol{a} = a_T \hat{\boldsymbol{T}} + a_N \hat{\boldsymbol{N}}. \tag{3.19}$$

Then we say the the **tangential component** of acceleration is $a_T \hat{\boldsymbol{T}}$ and the **normal component** of acceleration is $a_N \hat{\boldsymbol{N}}$.

For computations, if you've already found the $\hat{\boldsymbol{T}}$ and $\hat{\boldsymbol{N}}$ vectors then it's easy to compute the components of acceleration by just taking dot products. Then

$$a_T = \boldsymbol{r}'' \cdot \hat{\boldsymbol{T}},$$
$$a_N = \boldsymbol{r}'' \cdot \hat{\boldsymbol{N}},$$

and then compute $a_T\hat{\boldsymbol{T}}$ and $a_N\hat{\boldsymbol{N}}$. This process can be sped up by noticing that after one computes the tangential component $a_T\hat{\boldsymbol{T}}$ (which is easier), then to obtain the normal component you can just use (3.19) and write $a_N\hat{\boldsymbol{N}} = \boldsymbol{r}'' - a_T\hat{\boldsymbol{T}}$. On the other hand, if one is only interested in the *magnitudes* of the components of acceleration, then it's probably easier to just use the form of a_T and a_N coming from (3.18). Next time we'll write them explicitly in terms of the parametrisation $\boldsymbol{r}$.

Example. Let's find the components of acceleration for the circular helix,

$$\boldsymbol{r}(t) = (a\cos t, a\sin t, bt),$$

where $a > 0$. Here we use the method where we first compute the tangential component and then subtract it from the acceleration to find the normal component. We compute $\hat{\boldsymbol{T}}$ as before:

$$\begin{aligned}\boldsymbol{r}'(t) &= (-a\sin t, a\cos t, b),\\ |\boldsymbol{r}'(t)| &= \sqrt{a^2+b^2}.\end{aligned}$$

Thus

$$\hat{\boldsymbol{T}} = \frac{\boldsymbol{r}'}{|\boldsymbol{r}'|} = \frac{1}{\sqrt{a^2+b^2}}(-a\sin t, a\cos t, b).$$

Also,

$$\boldsymbol{r}''(t) = (-a\cos t, -a\sin t, 0),$$

so we can find the magnitude of the tangential component of acceleration by taking the dot product:

$$a_T = \boldsymbol{r}''(t)\cdot\hat{\boldsymbol{T}} = \begin{pmatrix}-a\cos t\\ -a\sin t\\ 0\end{pmatrix}\cdot\frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix}-a\sin t\\ a\cos t\\ b\end{pmatrix} = 0.$$

Thus there is no acceleration in the direction of the tangent! You should verify that by differentiating the speed $|\boldsymbol{r}'(t)|$ we get the same result. It then follows that $\boldsymbol{a}$ points in the direction of the unit normal vector. Therefore we immediately get (since $\boldsymbol{a} = a_T\hat{\boldsymbol{T}} + a_N\hat{\boldsymbol{N}}$) that the normal component of acceleration is

$$a_N\hat{\boldsymbol{N}} = (-a\cos t, -a\sin t, 0).$$

By considering the length of this vector we find in particular that

$$a_N = \sqrt{a^2\cos^2 t + a^2\sin^2 t + 0} = a.$$

Notice that this also gives us a new way of computing $\hat{\boldsymbol{N}}$ and the curvature in general. We have

$$a_N\hat{\boldsymbol{N}} = \boldsymbol{r}'' - a_T\hat{\boldsymbol{T}} = \boldsymbol{r}'',$$

since $a_T = 0$ in our case. Substituting in the definition of a_N gives

$$\begin{aligned}\kappa|\boldsymbol{r}'(t)|^2\hat{\boldsymbol{N}} &= \boldsymbol{r}''\\ \kappa(a^2+b^2)\hat{\boldsymbol{N}} &= (-a\cos t, -a\sin t, 0).\end{aligned}$$

By comparing the lengths of the vectors on each side lets us compute the curvature κ and by comparing the directions (so by normalising[1] the right-hand side) lets us compute $\hat{\boldsymbol{N}}$! So we get

$$\kappa(a^2+b^2) = \sqrt{a^2},$$

[1] Normalising a vector means dividing it by its length, i.e. converting it into a unit vector.

that is
$$\kappa = \frac{a}{a^2 + b^2}.$$
Also
$$\hat{\boldsymbol{N}} = \frac{1}{a}(-a\cos t, -a\sin t, 0) = (-\cos t, -\sin t, 0),$$
which agrees with what we've computed before. Notice that we could then compute $\hat{\boldsymbol{B}}$ through the cross product $\hat{\boldsymbol{B}} = \hat{\boldsymbol{T}} \times \hat{\boldsymbol{N}}$ to obtain the Frenet frame for the helix.

The only quantity we haven't yet computed for the helix is the torsion. You could do this directly from the definition by using the arc length parametrisation we found. Here's an alternative way (that works more generally).

Example. Let, as before,
$$\boldsymbol{r}(t) = (a\cos t, a\sin t, bt),$$
for $a > 0$. Recall that the torsion is defined through the derivative of the unit bi-normal as
$$\frac{d\hat{\boldsymbol{B}}}{ds} = -\tau\hat{\boldsymbol{N}}.$$
We've computed the unit bi-normal before as
$$\hat{\boldsymbol{B}} = \frac{1}{\sqrt{a^2 + b^2}}(b\sin t, -b\cos t, a).$$
We can then use the Chain Rule and the relation $\frac{ds}{dt} = |\boldsymbol{r}'|$ to obtain
$$\begin{aligned} \frac{d\hat{\boldsymbol{B}}}{ds} &= \frac{d\hat{\boldsymbol{B}}}{dt}\frac{dt}{ds} \\ &= \frac{\frac{d\hat{\boldsymbol{B}}}{dt}}{|\boldsymbol{r}'(t)|} \\ &= \frac{1}{a^2 + b^2}(b\cos t, b\sin t, 0). \end{aligned}$$
Now, since $\hat{\boldsymbol{N}} = (-\cos t, -\sin t)$ (see the previous example), we can read off the torsion (careful with the minus signs!) as
$$\tau = \frac{b}{a^2 + b^2}.$$
As an exercise you should check that the formula for torsion,
$$\tau = \frac{(\boldsymbol{r}' \times \boldsymbol{r}'') \boldsymbol{\cdot} \boldsymbol{r}'''}{|\boldsymbol{r}' \times \boldsymbol{r}''|^2},$$
gives the same result.

Let's do an example of finding the *magnitudes* of components of acceleration, this time by using the formulae for $\boldsymbol{r}$.

Example. Let
$$\boldsymbol{r}(t) = (\sqrt{2}t, e^t, e^{-t}).$$
We want to find the magnitudes of components of acceleration. We can rewrite the magnitude of the tangential component from (3.18) in terms of the parametrisation $\boldsymbol{r}$ as
$$\boxed{a_T = \frac{d}{dt}|\boldsymbol{r}'(t)|.}$$
In order to use this we need to compute the speed:
$$\boldsymbol{r}'(t) = (\sqrt{2}, e^t, -e^{-t}),$$
$$|\boldsymbol{v}(t)| = |\boldsymbol{r}'(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}.$$
It follows then immediately that
$$a_T = \frac{d}{dt}|\boldsymbol{r}'(t)| = e^t - e^{-t}.$$
For the magnitude of the normal component we know that $a_N = |\boldsymbol{v}(t)|^2\kappa$. By using (3.12) we can rewrite this in an useful form in terms of $\boldsymbol{r}$:
$$\boxed{a_N = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{|\boldsymbol{r}'|}.} \tag{3.20}$$
We compute
$$\boldsymbol{r}''(t) = (0, e^t, e^{-t}),$$
so that
$$\boldsymbol{r}' \times \boldsymbol{r}'' = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \sqrt{2} & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{vmatrix} = \begin{pmatrix} 2 \\ -\sqrt{2}e^{-t} \\ \sqrt{2}e^t \end{pmatrix}.$$
We compute the length
$$|\boldsymbol{r}' \times \boldsymbol{r}''| = \sqrt{4 + 2e^{-2t} + 2^{2t}} = \sqrt{2}\sqrt{(e^t + e^{-t})^2} = \sqrt{2}(e^t + e^{-t}).$$
Therefore, substituting into (3.20) we get
$$a_N = \frac{\sqrt{2}(e^t + e^{-t})}{e^t + e^{-t}} = \sqrt{2}.$$
We've thus computed the *magnitudes* of the components of acceleration. If we wanted to find the actual (vector) components of acceleration, we would still need to compute $\hat{\boldsymbol{T}}$ and $\hat{\boldsymbol{N}}$ so that we could form $a_T\hat{\boldsymbol{T}}$ and $a_N\hat{\boldsymbol{N}}$. You can do this but it's a bit inefficient and for such questions I would recommend you to proceed just as we did in the first example (by computing $a_T\hat{\boldsymbol{T}}$ and then subtracting it from $\boldsymbol{a}$).

Chapter 4

Multivariable Functions and Their Extrema

So far all of our functions have depended on one variable only. If we want to understand three-dimensional space properly, we need to consider functions such as $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x,y,z) = (x^2y, ye^z + x, xy)$ and in particular to be able to do calculus with such functions. This is important because we saw, for example, that a function $g : D \to \mathbb{R}$, with the domain $D \subseteq \mathbb{R}^2$, describes a surface in three dimensions by setting $z = g(x,y)$. You should now read section §14.1 in the book, which has examples of visualising functions of this type (we already covered the basics back when we first introduced surfaces). Recommended problems from that section are:

Recommended problems from §14.1: **61–66**.

For now we focus on the simpler case of functions mapping from $\mathbb{R}^n$ to $\mathbb{R}$, but everything generalises to the case of $f : \mathbb{R}^n \to \mathbb{R}^m$ in a straightforward way (at least until we introduce derivatives). A function mapping to $\mathbb{R}$, for example $f(x,y) = xy^2$, is called a **scalar function**.

4.2 Limits and Continuity

Recommended problems from §14.2: **11, 14, 15, 16**.

The first step towards developing calculus is to understand limits of functions. For functions of more than one variable the situation is much more complicated compared to the one variable case. This is because with one variable (say $\lim_{x\to a} g(x)$) there are only two ways for x to approach a: from above and below (the corresponding limits are denoted by $\lim_{x\to a^+} g(x)$ and $\lim_{x\to a^-} g(x)$, respectively), as can be seen in Figure 4.1. On the other hand, in the higher variable case we have infinitely many directions to approach the limit point! Consider $f(x,y)$ with (x,y) approaching the point $(0,0)$,

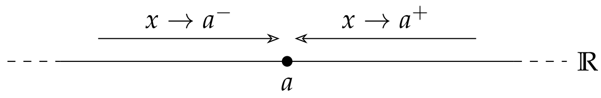


Figure 4.1: Two ways to approach a limit in one dimension.

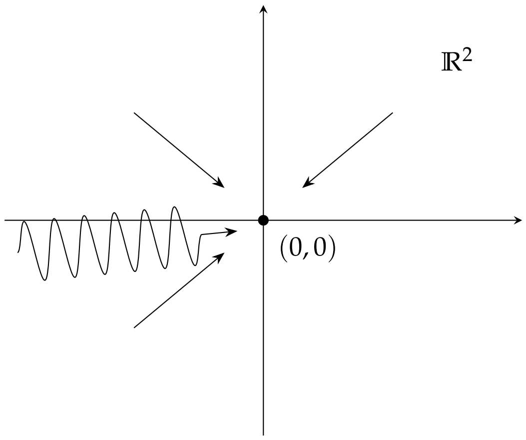


Figure 4.2: Infinitely many ways to approach the limit point.

we can approach the origin through any line or even through any curve, as shown in Figure 4.2. The idea is that the limit has to be the same no matter how we approach the point $(0,0)$. Let's make this more precise.

Definition 4.1. Let $f : D \to \mathbb{R}$ be a function with domain $D \subseteq \mathbb{R}^2$ and fix a point $(a,b) \in D$. We say that the **limit** of $f(x,y)$, as (x,y) approaches (a,b), is L if for every $\epsilon > 0$ we can find $\delta > 0$ such that if $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, and $(x,y) \in D$, then $|f(x,y) - L| < \epsilon$. We denote this by

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

Essentially what this definition is saying is that if you give me the size of an error (ϵ is the error between the actual limit L and the value $f(x,y)$), then I have to be able to give you a radius $\delta > 0$ such that for any point (x_0, y_0) within a radius δ from (a,b) the value of $f(x_0, y_0)$ is within the error ϵ from the limit L. Make sure you understand how this description follows from the definition.

Example. Let

$$f(x,y) = \frac{xy}{x^2 + y^2}.$$

Does the limit of $f(x,y)$ as $(x,y) \to (0,0)$ exist? Notice that while f is not defined at $(0,0)$, the limit could still very well exist (if you look at the definition of the limit, we never have to consider the case when $(x,y) = (a,b)$). A way to prove that a limit doesn't exist is to find two different directions such that the limit along them is different. This contradicts the definition of the limit (since it would have to be equal from any direction) so we can conclude that the limit doesn't exist.

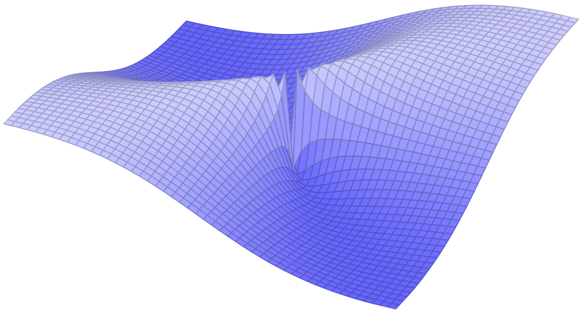


Figure 4.3: The surface $f(x,y) = \dfrac{xy}{x^2+y^2}$.

Let's see what happens when we approach the origin along the x-axis. To do this we set $(x,y) = (t,0)$ and look at $t \to 0$. Then

$$f(t,0) = \frac{t \cdot 0}{t^2+0^2} = 0.$$

Since this is a constant function, the limit as $t \to 0$ is 0. Now try the y-axis, $(x,y) = (0,t)$, as $t \to 0$. Then $f(0,t) = 0$. Since this is what we had before, it is not helpful to us! Let's make an educated guess and try approaching along a diagonal line $x = y = t$. Then

$$f(t,t) = \frac{t^2}{2t^2} = \frac{1}{2},$$

and we've won! We managed to find two directions such that the limit along them is distinct. Thus we conclude that $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist. In Figure 4.3 we can see the discontinuity at $(0,0)$ clearly (that's where the plotting software fails!).

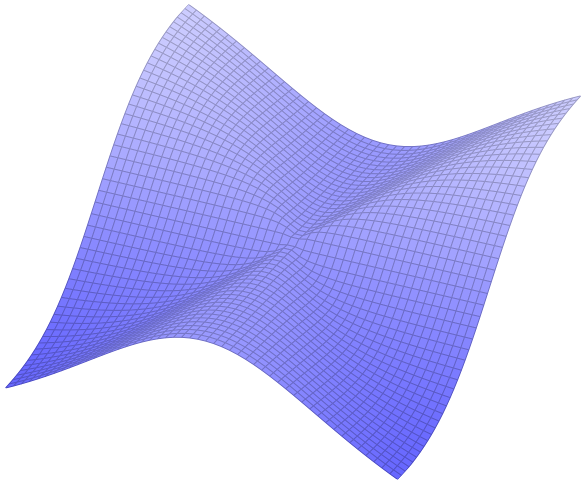


Figure 4.4: The surface $f(x,y) = \dfrac{x^2y}{x^2+y^2}$.

For limits of multivariable (scalar) functions all the basic rules of single variable limits generalise. For example Algebra of Limits and the Sandwich theorem still apply. We'll next see an example of how to prove that a limit actually exist. In the lectures I did it with the Sandwich theorem, but here I will first show how to do it with the ϵ–δ definition of the limit.

Example. Let

$$f(x,y) = \frac{x^2y}{x^2+y^2}.$$

Let's prove that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

Fix $\epsilon > 0$. We need to find $\delta > 0$ such that if $0 < \sqrt{x^2+y^2} < \delta$ then $|f(x,y) - 0| < \epsilon$. Now

$$\left|\frac{x^2y}{x^2+y^2}\right| = |y|\frac{x^2}{x^2+y^2}, \tag{4.1}$$

but $y^2 \geq 0$ so $x^2 \leq x^2+y^2$. It follows that

$$\frac{x^2}{x^2+y^2} \leq 1.$$

Substituting this back to (4.1) gives

$$|f(x,y)| = \left|\frac{x^2y}{x^2+y^2}\right| \leq |y|.$$

Also we notice that $|y| \leq \sqrt{x^2+y^2} < \delta$. So if we simply pick $\delta = \epsilon$ then it follows that $|f(x,y)| < \epsilon$, which concludes the proof. In Figure 4.4 you can see how the graph of f approaches the origin from all directions.

Let's now see the more straightforward way of proving the existence of the limit with the Sandwich theorem.

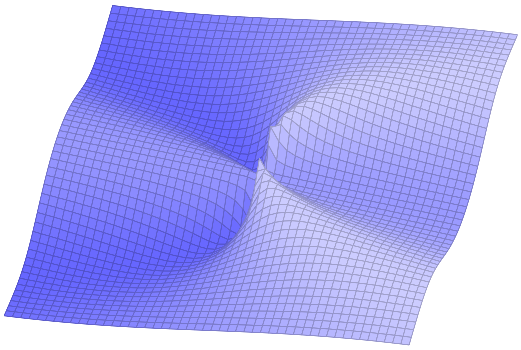


Figure 4.5: The surface $f(x,y) = \dfrac{xy^2}{x^2+y^4}$.

Example. Again, let

$$f(x,y) = \frac{x^2y}{x^2+y^2}.$$

We observed before that

$$0 \leq \frac{x^2}{x^2+y^2} \leq 1.$$

Since $-|y| \leq y \leq |y|$, we get

$$-|y| \leq \frac{x^2y}{x^2+y^2} \leq |y|.$$

But as $(x,y) \to (0,0)$ also $\pm|y| \to 0$. Thus by the Sandwich theorem also

$$\frac{x^2y}{x^2+y^2} \to 0,$$

as $(x,y) \to (0,0)$.

We can now define continuous functions analogously to the one variable setting.

Definition 4.2. Let f be as before (in Definition 4.1). If

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b),$$

then we say that f is **continous** at (a,b).

An important point to emphasise is that even if a limit exists (and is equal) along all alines, this is *not* enough to say with certainty that the full limit exists. Thus considering different directions is not an effective way to prove the existence of a limit, only the non-existence. The next example illustrates this.

Example. Let

$$f(x,y) = \frac{xy^2}{x^2+y^4}.$$

Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? Looking along the axes we see

$$f(t,0) = 0,$$
$$f(0,t) = 0.$$

This is not very helpful. In fact, if we consider the limit along any line $y = mx$, that is $(x, y) = (t, mt)$, then

$$f(t, mt) = \frac{m^2t^3}{t^2 + m^4t^4}.$$

We want to compute the limit of this fraction as $t \to 0$. This is easily done as before by dividing the numerator and denominator by t^2:

$$\frac{m^2t^3}{t^2 + m^4t^4} = \frac{m^2t}{1 + m^4t^2}.$$

But then the limit of the denominator is just 1 so Algebra of Limits applies and we see that

$$\frac{m^2t}{1 + m^4t^2} \to 0,$$

as $t \to 0$. We conclude that $f(x, y)$ tends to 0 if we approach the origin along any line. This, however, is not enough to say that the limit exists! Indeed, let's approach the origin through the parabola $y^2 = x$, that is $(x, y) = (t^2, t)$. Then

$$f(t^2, t) = \frac{t^2t^2}{t^4 + t^4} = \frac{1}{2}.$$

Thus we see that

$$f(t,0) \to 0,$$
$$f(t^2, t) \to \frac{1}{2},$$

as $t \to 0$, so the limit does not exist. You can see this behaviour in Figure 4.5.

It is also possible to use polar coordinates to evaluate limits of scalar functions of the form $f(x, y)$. This is especially useful if the function contains factors of the type $x^2 + y^2$ in which case the trigonometric functions disappear. In polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$, the limit $(x, y) \to 0$ corresponds to $r \to 0^+$.

Example. Let

$$f(x, y) = (x^2 + y^2)\ln(x^2 + y^2).$$

We want to find the limit of this function as $(x, y) \to (0, 0)$. First writing f in terms of polar coordinates we get

$$f(r\cos\theta, r\sin\theta) = r^2 \ln r^2 = 2r^2 \ln r,$$

by the rules of the logarithm. Then we can compute the limit as

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} f(x, y) &= \lim_{r\to 0^+} f(r\cos\theta, r\sin\theta) \\
&= \lim_{r\to 0^+} 2r^2 \ln r \\
&= \lim_{r\to 0^+} -2\frac{\ln r}{-1/r^2}.
\end{aligned}$$

Notice that this limit is of the type $-\infty/-\infty$ so we can apply L'Hôpital's Rule:

$$\begin{aligned}\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{r\to 0^+} -2\frac{1/r}{2/r^3}\\ &= \lim_{r\to 0^+} -r^2\\ &= 0.\end{aligned}$$

4.3 Partial Derivatives

Recommended problems from §14.3: **41, 42, 43, 44**.

In order to understand how multivariable functions behave, we need to understand their rate of change when we move in different directions. The simplest case is when we consider the rate of change in the direction of only one of the variables while keeping others fixed. This is the idea of partial differentiation. Formally we have the following definition.

Definition 4.3. Again, let $f : D \to \mathbb{R}$ be a function with domain $D \subseteq \mathbb{R}^2$ and fix $(a,b) \in \mathbb{R}^2$. The **partial derivative** of f with respect to x at (a,b) is given by the limit (if it exists)

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}, \tag{4.2}$$

and the partial derivative with respect to y is given analogously by

$$\frac{\partial f}{\partial y}(a,b) = \lim_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h}.$$

There's a plethora of notation for partial derivatives. Here we list some of the more common ones. All of the following mean the same thing[1]:

$$\frac{\partial f}{\partial x},\ f_x,\ D_1 f,\ \partial_x f.$$

Similarly for the second variable

$$\frac{\partial f}{\partial y},\ f_y,\ D_2 f,\ \partial_y f$$

all have the same meaning. The quantities f_x and f_y are called the *first partial derivatives* of f. We can take further partial derivatives to obtain the *second partial derivatives* of f:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2},$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x}, \qquad f_{yx} = \frac{\partial^2 f}{\partial x \partial y}.$$

[1]Careful! Some sources use $f_1(x,y)$ to denote the partial derivative of f with respect to the first variable. We will never do this.

Here the derivatives on the second line are called *mixed partial derivatives* (as they contain partial derivatives with respect to more than one variable). Be careful when using the subscript notation! f_{xy} means "*differentiate f with respect to x first, then with respect to y*", so we read the variables in the opposite direction as with the standard notation. A simple way to think of partial differentation is the following: *differentiate with respect to one variable as usual, while treating all other variables as constant*. This applies as long as there is no interdependence between the variables (if there is, we need the Chain Rule).

Example. Let

$$f(x,y) = 2xy^3 + y + x^2y.$$

Then the first partial derivatives of f are

$$\frac{\partial f}{\partial x} = 2y^3 + 2xy,$$
$$\frac{\partial f}{\partial y} = 6xy^2 + 1 + x^2.$$

What about the mixed second partial derivatives? We get

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(6xy^2 + 1 + x^2) = 6y^2 + 2x.$$

On the other hand,

$$\frac{\partial^2 f}{\partial y \partial x} = 6y^2 + 2x.$$

So the mixed partial derivatives are equal! Is this a coincidence? The answer is no. In fact, this almost always happens (i.e. for nice enough functions).

The next theorem tells us exactly when the mixed partial derivatives are equal.

Theorem 4.4 (Clairaut's Theorem)**.** *Let f be as before. Suppose the mixed second partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist and are continuous in an open neighbourhood of (a,b). Then the second mixed partial derivatives are equal at (a,b), that is,*

$$\frac{\partial^2 f}{\partial x \partial y}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b).$$

This theorem immediately implies exchanging of higher order mixed partial derivatives too (as long as the continuity property is satisfied), e.g. if we look at $g(x,y,z)$ then

$$\frac{\partial^4 g}{\partial^2 x \partial y \partial z} = \frac{\partial^4 g}{\partial y \partial z \partial x^2} = \frac{\partial^4 g}{\partial x \partial z \partial y \partial x},$$

and so on. In summary: any order is fine. Let's see an example in which the above theorem does *not* work.

Example. Let

$$f(x,y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

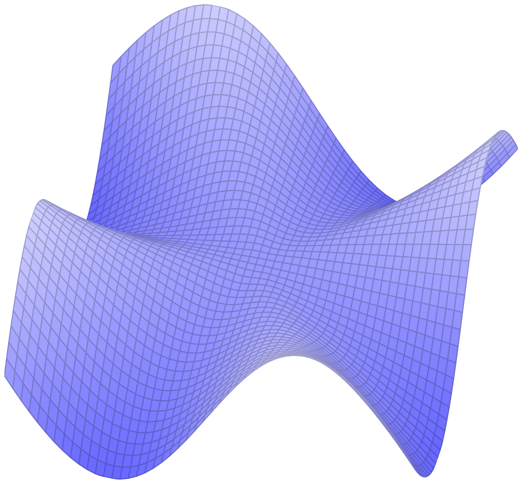


Figure 4.6: The surface $f(x,y) = \dfrac{x^3y - xy^3}{x^2+y^2}$.

As an exercise you should prove that this function is continuous (you can do it with the same Sandwich trick that we saw before). You can see the behaviour around the origin in Figure 4.6. Let's compute the first partial derivatives. If $(x,y) \neq (0,0)$, we can just use the formula in the definition to get

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= \frac{(3x^2y - y^3)(x^2+y^2) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2} \\ &= \frac{y}{(x^2+y^2)^2}(x^4 + 4x^2y^2 - y^4).\end{aligned}$$

At $(0,0)$ we need to use Definition 4.2. This gives

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

It's then possible to see that f_x is continuous everywhere (you should prove this!). Similarly for the partial derivative with respect to y we find

$$\begin{aligned}\frac{\partial f}{\partial y}(x,y) &= \frac{x}{(x^2+y^2)^2}(x^4 - 4x^2y^2 - y^4), && (\text{if } (x,y) \neq (0,0)) \\ \frac{\partial f}{\partial y}(0,0) &= 0.\end{aligned}$$

This is also continuous. You can differentiate one more time to see that away from $(0,0)$ both of the mixed partial derivatives are continuous and thus equal by Clairaut's Theorem. At $(0,0)$ we have to use the definition:

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(0,h) - \frac{\partial f}{\partial x}(0,0)}{h} = \frac{-h}{h} = -1.$$

Similarly,

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h} = \frac{h}{h} = 1.$$

Thus the mixed partial derivatives do not agree at the origin! Why does this not contradict Clairaut's Theorem? Well, if you computed the mixed second partial derivatives away from $(0, 0)$, then you could show that they are not continuous there, and thus Clairaut's Theorem does not apply.

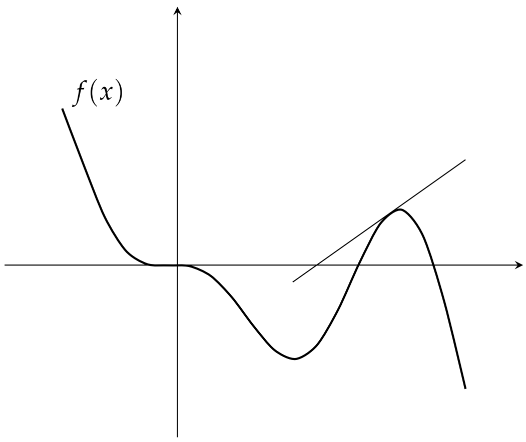


Figure 4.7: A single-variable function and its tangent line.

4.4 Linear Approximations and the Tangent Plane

Recommended problems from §14.4: **1, 2, 3, 4, 5, 6**.

In one variable setting we know that a differentiable function $f(x)$ is indistinguishable from its tangent line, provided we look closely enough at the point of differentiation. This means that at a point $x = a$ we can write

$$f(x) = \underbrace{f(a) + f'(a)(x-a)}_{\text{tangent line}} + \text{error}.$$

This is what we call a linear approximation. To see this, we start from the definition of the derivative

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

We can rearrange this by moving everything to the same side of the equation to get

$$\left| \lim_{x \to a} \frac{f(x) - f(a)}{x - a} - f'(a) \right| = 0,$$

where we have also taken absolute values. Then by factoring out the denominator we see that this is the same as

$$|x-a|^{-1} \underbrace{|f(x) - f(a) - f'(a)(x-a)|}_{\text{error in approximation}} \to 0.$$

So essentially we are saying that

$$\frac{|\text{error}|}{|x-a|} \to 0 \quad \text{as } x \to a.$$

This means that the error has to decay faster than linearly (which it does if f is differentiable!).

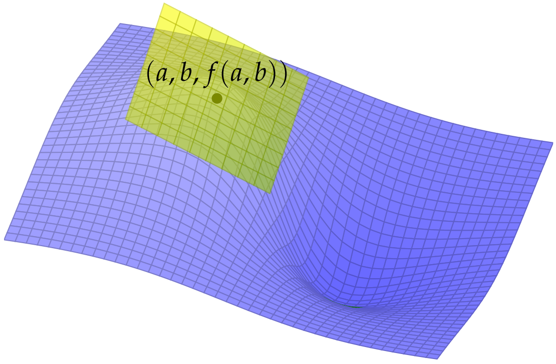


Figure 4.8: A surface and its tangent plane.

What about $f(x, y)$? To get a linear approximation we need to consider the change in both x and y variables. This leads to

$$f(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + \text{error}.$$

Geometrically the right-hand side gives us a tangent plane to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$. More precisely, the tangent plane to $z = f(x, y)$ when $(x, y) = (a, b)$ is given by

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b). \tag{4.3}$$

This is visualised in Figure 4.8. We'll now use the analogy between the tangent line and one variable functions on one hand, and the tangent plane and $f(x, y)$ on the other, to write down a definition for the derivative of $f(x, y)$. That is, we want to say that *if* the tangent plane approximates the function well at some point, *then* the function is differentiable there. Note that in this case the distance of (x, y) from (a, b) is given by $\sqrt{(x - a)^2 + (y - b)^2}$ (compare this with the one variable derivative). Now, let $x - a = h$ and $y - b = k$, then we arrive at the following definition.

Definition 4.5. We say $f(x, y)$ is **differentiable** at (a, b) if

$$\lim_{(h,k)\to(0,0)} \frac{|f(a + h, b + k) - f(a, b) - hf_x(a, b) - kf_y(a, b)|}{\sqrt{h^2 + k^2}} = 0.$$

An important point to make is that even if both of f_x and f_y exist, it is *not* enough to say that the derivative exists. Later we'll see what extra condition is needed. The above definition is concrete, but annoying to generalise to higher dimensions. Let's rewrite it in a different form. Let $\boldsymbol{h} = \binom{h}{k}$, $\boldsymbol{a} = \binom{a}{b}$ and (to spoil the idea) $f'(\boldsymbol{a}) = \begin{pmatrix} f_x(\boldsymbol{a}) & f_y(\boldsymbol{a}) \end{pmatrix}$. Then f is differentiable at $\boldsymbol{a}$ if

$$\frac{|f(\boldsymbol{a} + \boldsymbol{h}) - f(\boldsymbol{a}) - f'(\boldsymbol{a})\boldsymbol{h}|}{|\boldsymbol{h}|} \to 0,$$

as $|\boldsymbol{h}| \to 0$. Here the factor $f'(\boldsymbol{a})\boldsymbol{h}$ is understood just as matrix multiplication! The matrix $f'(\boldsymbol{a})$ is called the **derivative** of f at $\boldsymbol{a}$.

We can generalise this form of the definition easily. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$, be a function of n variables $\boldsymbol{x} = (x_1, \ldots, x_n)$. We can write f as $f(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \ldots, f_m(\boldsymbol{x}))$, where each of the f_i's is a scalar function. In full generality, the derivative of f is a *linear map* $f'(\boldsymbol{x})$ which satisfies

$$\frac{|f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) - f'(\boldsymbol{x})\boldsymbol{h}|}{|\boldsymbol{h}|} \to 0,$$

as $|\boldsymbol{h}| \to 0$. Now let's recall from your linear algebra course that linear maps correspond (not one-to-one) to matrices. So we can think of $f'(\boldsymbol{x})$ as a matrix! This matrix is called the **Jacobian matrix** of f and it is given explicitly as

$$f'(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}. \tag{4.4}$$

This is a $m \times n$ matrix in which we have simply just collected all the partial derivatives of f in a specific order. We can now give a precise condition for f to be differentiable at $\boldsymbol{a}$ based on its partial derivatives.

Theorem 4.6. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ with $f = (f_1, \ldots, f_m)$, where $f_1, \ldots, f_m$ are scalar functions of the variables $x_1, \ldots, x_n$. If all the partial derivatives $\frac{\partial f_i}{\partial x_j}$ exist and are continuous at $\boldsymbol{a} \in \mathbb{R}^n$, for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$, then f is differentiable at $\boldsymbol{a}$ with the derivative given by the Jacobian matrix $f'(\boldsymbol{a})$.*

It is a good idea to try to write out this theorem in the case of a scalar function $f(x, y)$.

Example. Let

$$f(x, y) = 2x^2 + y^2.$$

We want to find the derivative of f and the tangent plane to $z = f(x, y)$ at $(1, 1, 3)$. It's easy to compute that

$$f_x(x, y) = 4x, \qquad f_y(x, y) = 2y.$$

Notice that both of these are continuous so the derivative of f exists and is given by the Jacobian matrix $f'(x, y) = (4x, 2y)$. At our initial point this becomes $f'(1, 1) = (4, 2)$. For the tangent plane we just substitute into equation (4.3) and get

$$\begin{aligned} z &= f(1, 1) + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) \\ &= 3 + 4(x - 1) + 2(y - 1). \end{aligned}$$

4.5 The Chain Rule and the Total Derivative

Recommended problems from §14.5: **7, 8, 9, 10, 49, 55, 58**.

Let's again start from the case of a function of one variable and see how to generalise the idea to the multivariate case. Suppose $y(x)$ is a function of x, and $x(t)$ is a function of the variable t. The usual chain rule then tells us that the derivative of y with respect to t is given by

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

You can also write this in the (maybe) more common form $y'(t) = y'(x)x'(t)$.

Now, as our first step to generalise this we consider a scalar function of n variables $f(x_1, \ldots, x_n)$, where $f : \mathbb{R}^n \to \mathbb{R}$ and each of the x_i's are functions of the variable t. The simple form of the chain rule then says that

$$\frac{df}{dt} = \frac{d}{dt} f(x_1(t), \ldots, x_n(t)) = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \ldots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}. \tag{4.5}$$

Here the quantity $\frac{df}{dt}$ is sometimes called the **total derivative** of f (simply because we are considering it as a function of all the variables it actually depends on).

Example. Let

$$f(x, y) = xe^y,$$

where both x and y are functions of t given by

$$x = \cos t, \qquad y = t \sin t.$$

We can now compute $\frac{df}{dt}$ in two different ways. First, we could just use the definitions of x and y in terms of t and subtitute in to f to get

$$f(x(t), y(t)) = (\cos t)e^{t \sin t},$$

and then differentiate with respect to t:

$$\frac{df}{dt} = \frac{d}{dt}\left((\cos t)e^{t \sin t}\right) = (-\sin t)e^{t \sin t} + (\cos t)e^{t \sin t}(\sin t + t \cos t).$$

Or we can apply the chain rule. For this we need the partial derivatives

$$\frac{dx}{dt} = -\sin t, \qquad \frac{dy}{dt} = \sin t + t \cos t.$$

The chain rule then reads

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= e^y(-\sin t) + xe^y(\sin t + t \cos t). \end{aligned}$$

If you then substitute for x and y, we get at the same result as before (as we should).

We're now ready (or as ready as one can be) to state the chain rule for general multivariable functions. The setup for this is a bit tedious, bear with me.

Chain Rule (General form)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function of n variables and let $x : \mathbb{R}^k \to \mathbb{R}^n$ be a function of k variables $t_1, \ldots, t_k$. We set $f = (f_1, \ldots, f_m)$ and $x = (x_1, \ldots, x_n)$, where each $f_i(x)$ and $x_j(t_1, \ldots, t_k)$ is a scalar function. We then look at $g = f \circ x$. The chain rule tells us that the derivative of g with respect to $t = (t_1, \ldots, t_k)$ is*

$$g'(t) = f'(x)x'(t).$$

In terms of the Jacobian matrices this reads

$$\begin{pmatrix} \frac{\partial g_1}{\partial t_1} & \cdots & \frac{\partial g_1}{\partial t_k} \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial t_1} & \cdots & \frac{\partial g_m}{\partial t_k} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_1}{\partial t_k} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_k} \end{pmatrix}. \tag{4.6}$$

Notice that if we just look at the first element in the matrix on the left-hand side then this is just the simple chain rule we saw before:

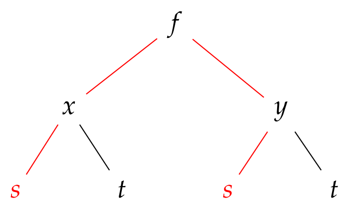
$$\frac{\partial g_1}{\partial t_1} = \frac{\partial f_1}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \ldots + \frac{\partial f_1}{\partial x_n}\frac{\partial x_n}{\partial t_1}.$$

Let's now do some examples of applying the chain rule. If we have a multivariable function, but we only want to find one of the partial derivatives, then it's often more efficient just to workout the chain rule (akin to the simple chain rule) for that particular case rather than computing the full Jacobians. To help with this we draw a tree diagram which shows the dependence of the function on each of its variables. We see how to do this in the next example.

Example. Let f be a scalar function of two variables given by $f(x,y) = ye^x$, where x and y are functions of s and t defined[2] as

$$x = st^2 \qquad\qquad y = s^2 t.$$

The tree diagram for f is the following:



We want to find $\frac{\partial f}{\partial s}$. To do this we write down the chain rule for this by reading off from the above diagram. First, look for all the nodes with s (in red). This tells us which terms to include in the chain rule as follows:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}.$$

We can then just substitute in the appropriate partial derivatives to get

$$\frac{\partial f}{\partial s} = ye^x t^2 + e^x(2st) = s^2t^3e^{st^2} + 2ste^{st^2}.$$

On the other hand, if we want to find all the partial derivatives (especially at a specific point), then it can be more efficient to compute the Jacobians.

Example. Let w be a function of x, y and z given by

$$w = xy + yz + zx,$$

where x, y and z are functions of r and θ defined by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = r\theta.$$

We want to find the Jacobian $\begin{pmatrix} \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \theta} \end{pmatrix}$ at the initial point $(r,\theta) = (2, \frac{\pi}{2})$.

At the initial point we have $(x,y,z) = (0,2,\pi)$. Let's compute the Jacobian matrices on the right-hand side of the chain rule. We get (at the initial point)

$$\begin{pmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} = \begin{pmatrix} y+z & z+x & x+u \end{pmatrix} = \begin{pmatrix} 2+\pi & \pi & 2 \end{pmatrix},$$

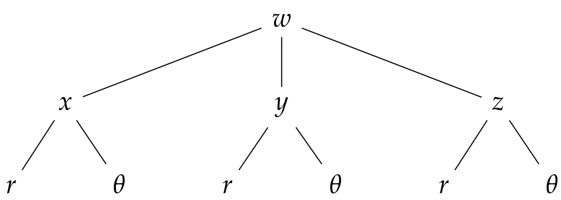
[2] To be pedantic, we should really write $x(s,t)$ and $y(s,t)$, but in the interest of avoiding clutter we suppress the dependence as it is clear from the context.

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ \theta & r \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 1 & 0 \\ \frac{\pi}{2} & 2 \end{pmatrix}.$$

So by the chain rule we have (at the initial point)

$$\begin{aligned} \left(\tfrac{\partial w}{\partial r} \quad \tfrac{\partial w}{\partial \theta}\right)\Big|_{\substack{r=2\\ \theta=\frac{\pi}{2}}} &= \begin{pmatrix} 2+\pi & \pi & 2 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ 1 & 0 \\ \frac{\pi}{2} & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2\pi & -2\pi \end{pmatrix}. \end{aligned}$$

Alternatively, we could also draw the tree diagram for this problem and work through it as in the previous example:



Implicit Differentiation

We will now talk about an important topic which is related to the chain rule. You might be familiar with the concept of implicit differentiation from your single variable calculus course. The basic idea is that we have a function defined implicitly in terms of its variable through and equation. The goal is to find derivatives of such functions. For example, suppose $y = y(x)$ is a function of x defined implicitly by the equation

$$y^2 + x^2 = 1. \tag{4.7}$$

We then recall that to differentiate this with respect to x we just get

$$2y\frac{dy}{dx} + 2x = 0$$

by the single variable chain rule. It is then possible to solve this for $\frac{dy}{dx}$ to get

$$\frac{dy}{dx} = \frac{-x}{y}. \tag{4.8}$$

There are a couple of points to make here. First, the equation (4.7) certainly doesn't define y as a function of x everywhere. Indeed, for example if $x = 2$ then there is no (real) value of y which satisfies the equation. So some care has to be taken about the domain of $y(x)$. Moreover, even if the original equation has a solution, then it doesn't guarantee that y will be a well-defined function. Take for example $x = 0$, y could then be either equal to 1 or -1, but such multi-valued relation does not define a function! A function has to have a unique value at each point. This is a bit of a moot point though, since the reason for defining functions implicitly is that there often is no single way to represent them in their domain. We can see this in the case of our example by looking at the picture Figure 4.9. We see that the observation $y(0) = \pm 1$ essentially just means

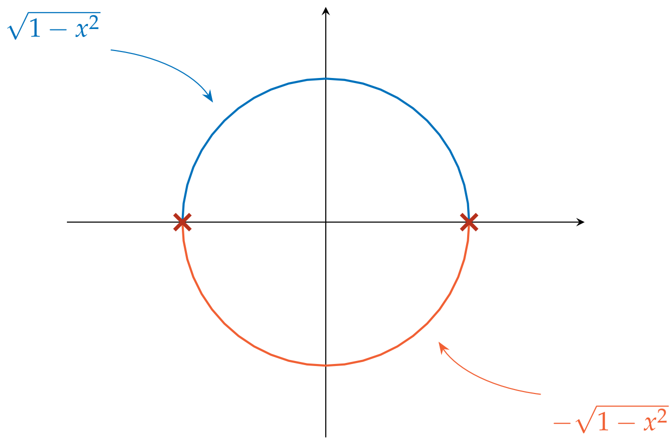
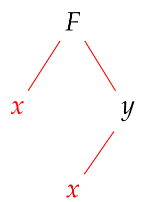


Figure 4.9: A circle as an implicit function y with the points where Implicit function theorem doesn't apply marked with crosses.

that at different points on the xy-plane the function has a different form. In other words, there is no unique formula that works everywhere. Our defines defines a circle and we know that the top part can be expressed as $y(x) = \sqrt{1-x^2}$ while the bottom part is $y(x) = -\sqrt{1-x^2}$, as we see in Figure 4.9. Now look at the point $(1,0)$. There we notice that if we want to move along the circle either in the clockwise or anti-clockwise direction we have to use different functions! So there is no single function that we can pick that works in the neighbourhood of the point $(1,0)$ (i.e. in both directions along the circle). Why does this happen? Can we predict when a solution does not exist?

To answer this, let's start by making things more precise and use the formalism of multivariable functions that we've developed. Let $F(x,y) = y^2 + x^2 - 1$, where we've just moved everything in (4.7) to one side of the equation and set that to be our function F (with the variable x as the first coordinate and the unknown function y as the second). So we're now considering solutions to the equation $F(x,y) = 0$. Let's find the derivative of this with respect to x. F has the following tree diagram:



Notice that there is dependence on x on two levels (in red). Differentiating $F(x,y) = 0$ with respect to x gives then (by the chain rule)

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\partial y \partial x = 0.$$

Here we could've not included the $\frac{\partial x}{\partial x}$ factor since it's just trivially equal to 1 (why?). Solving the above equation gives

$$\frac{\partial y}{\partial x} = \frac{-F_x}{F_y}, \tag{4.9}$$

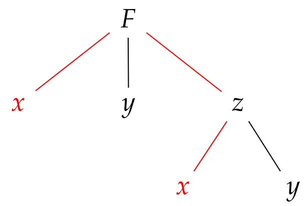
as long as $F_y \neq 0$. Now let's go back to the situation before. We saw that something suspicious happens when $(x, y) = (1, 0)$. However, we immediately notice that $F_y(1, 0) = 0$, so that (4.9) is not well-defined! This turns out to be the crucial condition. A full answer is then provided by the Implicit function theorem. In our simple case it says the following: if $F(x, y) = 0$ has a solution, say, $(x, y) = (a, b)$ and furthermore if $F_y(a, b) \neq 0$, *then* in a neighbourhood of (a, b), y can (in principle) be solved for x (in other words, y defines a function of x), so that we have $F(x, y(x)) = 0$ in a neighbourhood of (a, b). Moreover, the derivative of this function $y(x)$ with respect to x is given by (4.9). An important point here is that the Implicit function theorem is quite abstract: it doesn't tell us *how* to solve y for x, it just tells us that in theory it *can* be done.

Let's look at another example with more than one variable.

Example. Suppose z is a function of x and y defined implicitly by

$$x^2 + 2y^2 + 3z^2 = 1.$$

Again, we set $F(x, y, z) = x^2 + 2y^2 + 3z^2 - 1$, where we put the variables x and y first in the list of arguments of F and the unknown z last. The tree diagram is



So if we differentiate $F(x, y, z) = 0$ with respect to x we get

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0.$$

Notice we didn't include the $\frac{\partial x}{\partial x}$ factor since it's just 1 and we didn't have to include the term with $\frac{\partial y}{\partial x}$ since y doesn't depend on x (as you can see from the tree diagram). We can then solve the above equation for

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}, \tag{4.10}$$

provided that $F_z \neq 0$. Similarly we get

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}. \tag{4.11}$$

Also, we notice that $F(x, y, z) = 0$ has a solution $(0, 0, 1/\sqrt{3})$ and that $F_z(0, 0, 1/\sqrt{3}) = 2\sqrt{3} \neq 0$. So in this case the Implicit function theorem tells us that in a neighbourhood of $(0, 0, 1/\sqrt{3})$, at least in principle, the unknown z can be solved for x and y, so that $F(x, y, z(x, y)) = 0$, with the partial derivatives of z given by (4.10) and (4.11).

So far we've only considered the case when we have a single unknown possibly in terms of multiple variables. But what happens if we have, say, n unknowns? Then we know that we need n equations to be able to determine them. So let's suppose we have

n unknowns $x_1, \dots, x_n$, all of which are functions of p variables $t_1, \dots, t_p$, given by the equations

$$\begin{aligned} F_1(t_1, \dots, t_p, x_1, \dots, x_n) &= 0, \\ &\vdots \\ F_n(t_1, \dots, t_p, x_1, \dots, x_n) &= 0. \end{aligned}$$

First, we form $F = (F_1, \dots, F_n)$. Then, as before, we differentiate the equation $F = 0$ with respect to $t = (t_1, \dots, t_p)$. To do this we need to compute the relevant Jacobian matrices. For F we get

$$F' = \left(\begin{array}{ccc|ccc} \frac{\partial F_1}{\partial t_1} & \dots & \frac{\partial F_1}{\partial t_p} & \frac{\partial F_1}{\partial x_1} & \dots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial F_n}{\partial t_1} & \dots & \frac{\partial F_n}{\partial t_p} & \frac{\partial F_n}{\partial x_1} & \dots & \frac{\partial F_n}{\partial x_n} \end{array}\right).$$
$$\underset{n\times p}{F_t} \qquad\qquad \underset{n\times n}{F_x}$$

Here we write the Jacobian matrix as an augmented matrix with the part with the partial derivatives with respect to t denoted by F_t and the part with partial derivatives with respect to x by F_x. That is, we have $F' = \left(F_t \mid F_x\right)$. For the chain rule we also need to differentiate the function we are composing F with, that is, the vector

$$\begin{pmatrix} t_1 \\ \vdots \\ t_p \\ x_1(t_1,\dots,t_p) \\ \vdots \\ x_n(t_1,\dots,t_p \end{pmatrix}.$$

This is a function of $t_1, \dots, t_p$ and the Jacobian is the $(p+n) \times p$ matrix

$$\left(\begin{array}{cccc} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 1 \\ \hline \frac{\partial x_1}{\partial t_1} & \dots & \dots & \frac{\partial x_1}{\partial t_p} \\ \vdots & & & \vdots \\ \frac{\partial x_n}{\partial t_1} & \dots & \dots & \frac{\partial x_n}{\partial t_p} \end{array}\right) = \left(\begin{array}{c} I_p \\ \hline x'(t) \end{array}\right),$$

where the top part of the matrix is just the $p \times p$ identity matrix, I_p since $\frac{\partial t_i}{\partial t_j} = 1$ if and only if $i = j$ and is 0 otherwise. We also put above $x'(t)$ in the part of the matrix corresponding to the Jacobian of $x = (x_1, \dots, x_n)$ with respect to $t_1, \dots, t_p$. Combining all of this gives us the derivative of $0 = F$ as

$$\begin{aligned} 0 &= \left(F_t \mid F_x\right) \left(\begin{array}{c} I_p \\ \hline x'(t) \end{array}\right) \\ &= \underset{n\times p}{F_t}\ \underset{p\times p}{I_p} + \underset{n\times n}{F_x}\ \underset{n\times p}{x'(t)}, \end{aligned}$$

by the chain rule. So we see that if F_x is invertible (that is if $\det F_x \neq 0$) then we can solve the above equation for $x'(t)$ as

$$x'(t) = -F_x^{-1} F_t. \tag{4.12}$$

The general Implicit Function Theorem then tells us that *if* $F = 0$ has a solution and $\det F_x \neq 0$ at that point, then in a neighbourhood of the initial point the unknowns $x_1, \ldots, x_n$ can be (in theory) be solved for $t_1, \ldots, t_n$ with all the partial derivatives of the x_i's given by (4.12). Next time we'll do an example of this more general implicit function theorem.

Let's do an example of using the general form of the implicit function theorem.

Example. Suppose we have two unknowns x and y which both depend on the variables s and t defined implicitly through the equations

$$\begin{aligned} f(s,t,x,y) &= -tx^2 + 3s^2xy - 2 = 0, \\ g(s,t,x,y) &= t^3xy + sy^3 - 2 = 0. \end{aligned}$$

Then set $F = (f,g)$ (to be precise, this means F is a function from $\mathbb{R}^4$ to $\mathbb{R}^2$ given by $F(s,t,x,y,z) = (f(s,t,x,y), g(s,t,x,y))$. We see that $F = (0,0)$ has a solution $(s,t) = (1,1)$ and $(x,y) = (1,1)$. To apply the implicit function theorem we need to compute the Jacobians. We have the Jacobian matrix of F as

$$\begin{pmatrix} f_s & f_t & f_x & f_y \\ g_s & g_t & g_x & g_y \end{pmatrix} = \left(\begin{array}{cc|cc} 6sxy & -x^2 & -2tx+3s^2y & 3s^2x \\ y^2 & 3t^2xy & t^3y & t^3x+3sy^2 \end{array}\right), \tag{4.13}$$

which we write as $\left(F_t \mid F_x\right)$ by using the notation from before. At the initial point we then get

$$\left(F_t \mid F_x\right)\big|_{(s,t,x,y)=(1,1,1,1)} = \left(\begin{array}{cc|cc} 6 & -1 & 1 & 3 \\ 1 & 3 & 1 & 4 \end{array}\right).$$

Next we have to check whether $F_x = \left(\begin{smallmatrix} 1 & 3 \\ 1 & 4 \end{smallmatrix}\right)$ is invertible. A matrix is invertible if and only if its determinant is non-zero. Recall that for a 2×2 matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, its determinant is given by

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc,$$

so in our case $\det F_x = 4 - 3 = 1 \neq 0$ so that F_x is invertible at the initial point. This means that the Implicit function theorem applies. The inverse of a 2×2 matrix is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and hence

$$F_x^{-1} = \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}$$

at the initial point. So therefore by the Implicit function theorem the unknowns x and y can be solved for s and t (so they determine functions of s and t) in a neighbourhood of the initial point, so $F(s,t,x(s,t),y(s,t)) = (0,0)$ near $(s,t,x,y) = (1,1,1,1)$. Moreover, at the initial point the partial derivatives of x and y with respect to t and s are given by

$$\begin{aligned} \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial y} \end{pmatrix} &= -\begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 6 & -1 \\ 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} -21 & 13 \\ 5 & -4 \end{pmatrix}. \end{aligned}$$

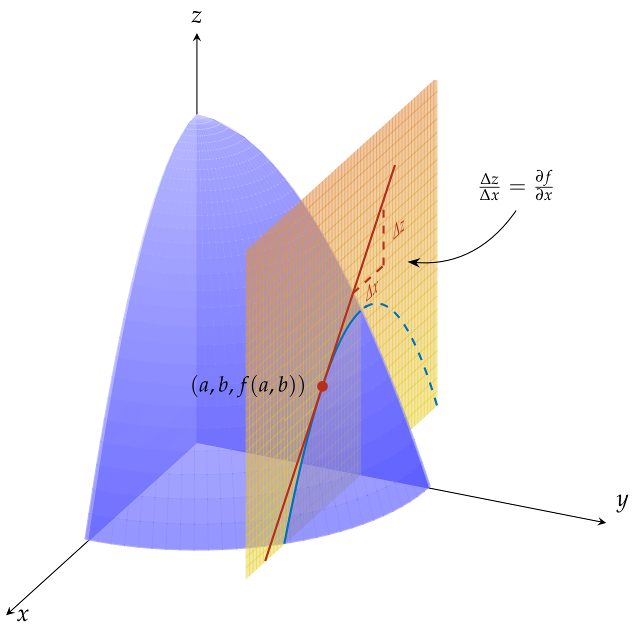


Figure 4.10: Partial derivatives give the slope of the tangent line in the x or y directions.

4.6 Gradient and Directional Derivatives

Recommended problems from §14.6: **11, 12, 13, 21, 22, 23**.

So far we've learnt how to compute the rate of change of $f(x, y)$ in the direction of its variables (the coordinate axes) by computing the partial derivatives of f with respect to x and y. If you look at the definition of partial derivatives (4.2) then you see that we are evaluating f on a line in the direction of the x axis, passing through the point (a, b). The slope of this line is exactly the value of the partial derivative at the point (a, b) (just like for functions of one variable!). In other words, if we intersect our surface $z = f(x, y)$ with the plane $y = b$ the intersection defines a curve which passes through the point $(a, b, f(a, b)$. The tangent vector of this curve is then necessarily also tangent to the surface (since the curve lies on the surface). You can see this in Figure 4.10.

So we know how to compute the rate of change of f in the directions of the coordinate axes and in particular we know how to find the slope of the tangent line in these directions. What about other directions? To answer this we'll introduce our first vector operator, the gradient. Everything in the upcoming discussion applies equally to functions of two variables just by ignoring the third coordinate.

Definition 4.7. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar function $f(x, y, z)$. The **gradient** of f is denoted by the nabla symbol ∇f and defined as the vector

$$\nabla f = \mathbf{grad}\, f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = f_x\, \boldsymbol{i} + f_y\, \boldsymbol{j} + f_z\, \boldsymbol{k}.$$

So suppose we want to compute the rate of change of $f(x,y,z)$ in the direction of the unit vector $\hat{\boldsymbol{u}} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$. Here (and whenever we talk about *direction*) it is important to make sure that $\hat{\boldsymbol{u}}$ actually is a *unit vector* so that $|\hat{\boldsymbol{u}}| = 1$. Otherwise we will get wrong answers. Now, to compute the desired rate of change, by analogy, we want to consider f along the line in the direction of $\hat{\boldsymbol{u}}$ (as opposed to along the lines in the direction of x or y axes). This line is of course given by

$$t \mapsto (x + tu_1, y + tu_2, z + tu_3).$$

Then to obtain the rate of change we just apply chain rule to get

$$\frac{\partial}{\partial t} f(x + tu_1, y + tu_2, z + tu_3) = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2 + \frac{\partial f}{\partial z} u_3 = \nabla f(x,y,z) \cdot \hat{\boldsymbol{u}},$$

where we have written the result in terms of the gradient of f.

Definition 4.8. Suppose $\boldsymbol{u}$ is a unit vector and $f(x,y,z)$ is a scalar function. Then the quantity

$$D_{\boldsymbol{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \boldsymbol{u},$$

is called the **directional derivative** of f in the direction $\boldsymbol{u}$.

We have now learnt how to compute the rate of change of f in any direction. The first natural question is in which direction is the rate of change (say, increase) the largest? To answer this we just have to apply the definition of the dot product in the definition of the directional derivative above. Let θ be the angle between ∇f and $\boldsymbol{u}$, then

$$\begin{aligned} \nabla f \cdot \boldsymbol{u} &= |\nabla f||\boldsymbol{u}| \cos\theta \\ &= |\nabla f| \cos\theta, \end{aligned}$$

since $|\boldsymbol{u}| = 1$. Clearly this is largest when $\cos\theta = 1$ so that $\theta = 0$, which means that $\boldsymbol{u}$ points in the same direction as ∇f (the direction is obtained by normalising to $\frac{\nabla f}{|\nabla f|}$). The actual maximal rate of increase is then just $|\nabla f|$. Similarly, the greatest rate of decrease is in the direction $-\frac{\nabla f}{|\nabla f|}$ (so opposite to ∇f) and the rate is $-|\nabla f|$. Also notice that if $\nabla f = 0$ then $D_{\boldsymbol{u}} f = 0$ in any direction $\boldsymbol{u}$.

Example. Let

$$f(x,y,z) = x^2yz - xyz^3.$$

We want to find the directional derivative of f at $(2,-1,1)$ in the direction $\boldsymbol{u} = (0, \frac{4}{5}, \frac{-3}{5})$. First thing to check is that $|\boldsymbol{u}| = 1$, which indeed is true, so that $\boldsymbol{u}$ gives a well-defined direction. Next we compute the gradient of f as

$$\nabla f(x,y,z) = (2xyz - yz^3, x^2z - xz^3, x^2y - 3xyz^2).$$

At $(2,-1,1)$ this becomes

$$\nabla f(2,-1,1) = (-3,2,2).$$

Thus the directional derivative is

$$D_{\boldsymbol{u}} f(2,-1,1) = (-3,2,2) \cdot \frac{1}{5}(0,4,-3) = \frac{2}{5}.$$

Tangent Planes Revisited

We saw before how to find the tangent plane of a surface $z = f(x,y)$. Recall that a tangent plane to this surface at $(a,b,f(a,b))$ is given by the formula

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

We can now rewrite this equation with our new notation in terms of the gradient. Let $F(x,y,z) = f(x,y) - z$, then

$$\nabla F(a,b,f(a,b)) \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} a \\ b \\ f(a,b) \end{pmatrix} \right) = 0.$$

Recalling the vector definition of a plane we see here that $\begin{pmatrix} x-a \\ y-b \\ z-f(a,b) \end{pmatrix}$ is a vector lying on the plane and in particular that ∇F is *normal* vector to the tangent plane $(a,b,f(a,b))$. Since this holds at every point on the surface, we can see that ∇F is normal to the surface $F(x,y,z) = 0$ (this is a *level surface* of the function F) at every point.

So this is what we knew from before. What about if we have a surface that is not defined explicitly as a function of x and y (like above), but implicitly through some scalar function $F(x,y,z)$ by setting $F(x,y,z) = k$, where $k \in \mathbb{R}$ is any constant. That is, we look at a level surface of the function F. Denote the surface by S. Well, we can still see that ∇F is normal to S at every point. Thus if (a,b,c) is a point on this surface (so $F(a,b,c) = k$) then the tangent plane to S at (a,b,c) is given by

$$\nabla F(a,b,c) \cdot \begin{pmatrix} x-a \\ y-b \\ z-c \end{pmatrix} = 0. \tag{4.14}$$

Notice here that $\nabla F(a,b,c)$ means that you *first* compute the gradient, and *then* evaluate it at $(x,y,z) = (a,b,c)$ (otherwise you would be taking derivatives of constants so you would trivially just get zero). Let's do an example of finding the tangent plane.

Example. Let

$$F(x,y,z) = x + y + z - e^{xyz},$$

and look at the level surface $F(x,y,z) = 0$ of F. Then the gradient

$$\nabla F(x,y,z) = \begin{pmatrix} 1 - yze^{xyz} \\ 1 - zxe^{xyz} \\ 1 - xye^{xyz} \end{pmatrix}$$

is normal to the surface at every point. Now, our equation $F = 0$ has a solution $(0,0,1)$, in other words the point $(0,0,1)$ lies on the surface. What is the tangent plane at this point? Well, the gradient at this point is $\nabla F(0,0,1) = (1,1,1)$ so by subtituting into (4.14) we get

$$\begin{aligned} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z-1 \end{pmatrix} &= 0 \\ \iff \qquad x + y + z &= 1. \end{aligned}$$

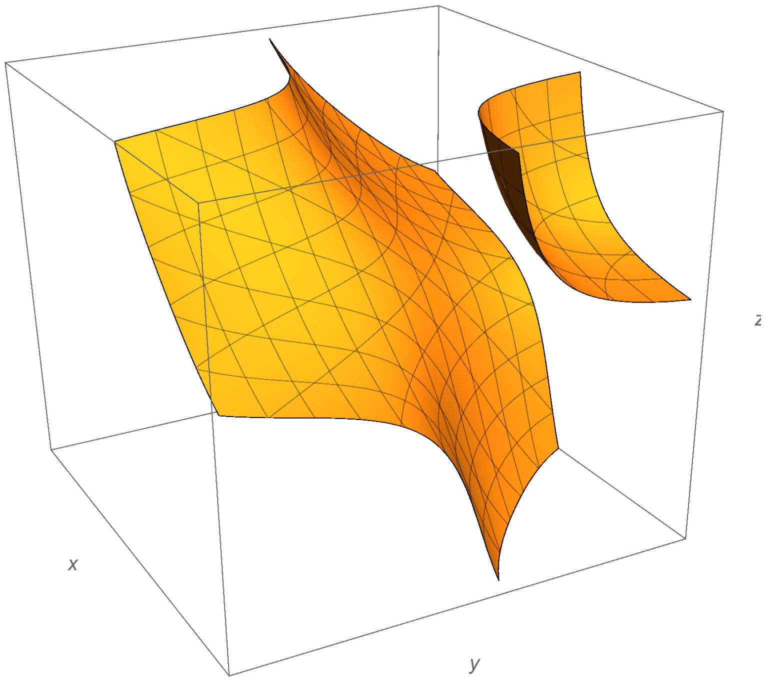


Figure 4.11: Surface defined implicitly by $x + y + z - e^{xyz} = 0$.

You can see the surface in Figure 1.10. Notice that since this surface is defined implicitly [3] it can have sheets on top of each other (compare this with the implicit equation of a circle we saw in 2D). These correspond to distinct explicit solutions to the equation in some subsets of the domain.

Let's summarise the most important properties of the gradient we've learnt so far. Suppose f is (any) scalar function. Then,

1) ∇f points in the direction of the greatest rate of increase of f,

2) the greatest rate of increase of f is then given by $|\nabla f|$,

3) ∇f is normal to the *level surfaces* of f (i.e. $f = \text{const.}$).

It's important not to confuse the *graph* of the function (recall a graph of a function $g(x,y)$ is the set $\{(x, y, g(x,y)) : x, y \in \mathbb{R}\}$) with its level surfaces/curves. For example, take the fuction $f(x,y) = x^2 + y^2$. The graph of this function is the surface $z = x^2 + y^2$ (so it is a three dimensional object), whereas the level curves of f are $x^2 + y^2 = k$, for different constants k, which are just circles of radius $\sqrt{k}$ on the xy-plane. Then, the gradient ∇f is perpendicular to the *level curves* of f, while $\nabla(f(x,y) - z)$ is perpendicular to the graph of f. This is illustrated in Figure 4.12.

Example. Let $\mathscr{C}$ be the curve given by the parametrisation

$$\boldsymbol{r}(t) = (1, t, 1 + t^2),$$

[3] Implicit functions are difficult to plot. While most of the graphics in these notes were produced directly in LaTeX with help from Ti*k*Z, `pgfplots`, and `gnuplot`, the pictures for implicit surfaces that do not have an easy parametrisation have been plotted in Mathematica.

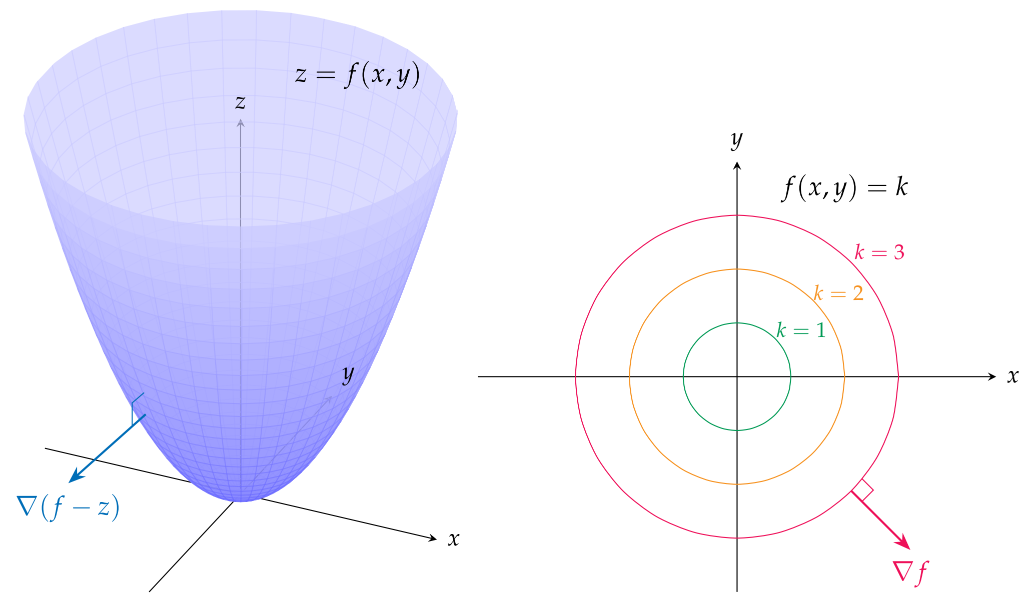


Figure 4.12: The graph and the level curves of the function $f(x,y) = x^2 + y^2$.

and let S be the surface given by the equation

$$x^2 + 2y + z = 5.$$

Let $F(x,y,z) = x^2 + 2y + z$ so that S is the level surface $F = 5$. Then the curve intersects S at exactly two points, P and Q, as you can see in 4.13. We want to find the angle between the tangent to $\mathscr{C}$ and the normal to S at each point of intersection, say, θ_1 and θ_2.

The first order of business is to find the coordinates of P and Q. To do this we just substitute the values of $\boldsymbol{r}$ into the equation for S to get

$$1^2 + 2t + (1 + t^2) = 5.$$

This factorises as

$$(t-1)(t+3) = 0,$$

so that our solutions are given by $t = 1$ and $t = -3$. Therefore the point P is given by $\boldsymbol{r}(1) = (1,1,2)$ and the point Q by $\boldsymbol{r}(-3) = (1,-3,10)$. Next we need the normal vector to S. This is given by the gradient:

$$\nabla(x^2 + 2y + z) = \begin{pmatrix} 2x \\ 2 \\ 1 \end{pmatrix},$$

so we see that at both P and Q, a normal vector is given by $\left(\begin{smallmatrix} 2 \\ 2 \\ 1 \end{smallmatrix}\right)$. Next we need a tangent vector to $\boldsymbol{r}$. This is of course given by

$$\boldsymbol{r}'(t) = (0, 1, 2t).$$

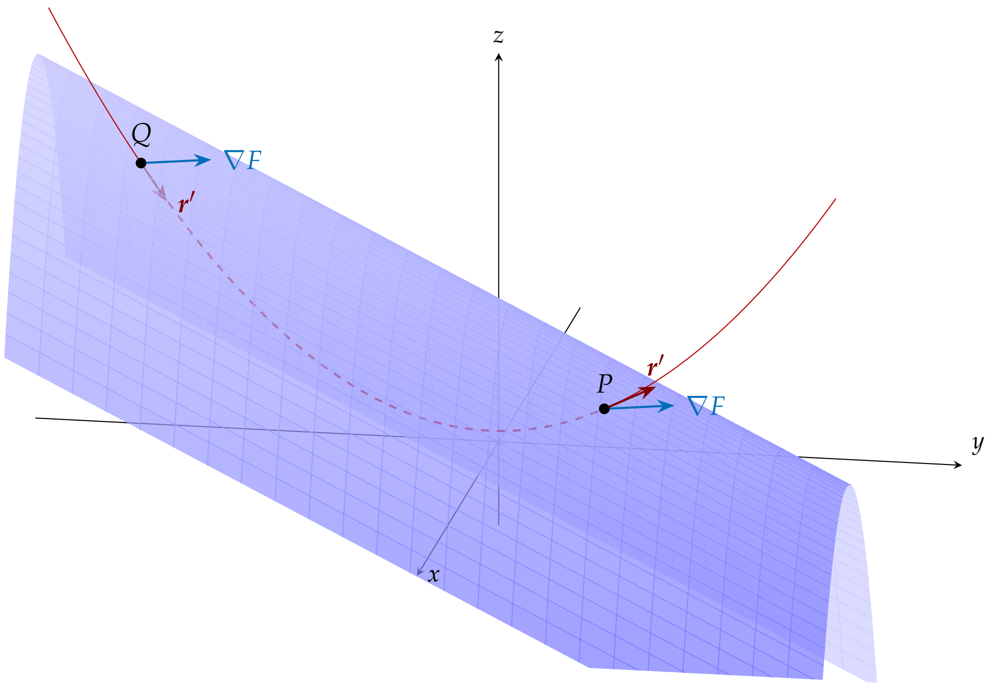


Figure 4.13: The curve $\mathscr{C}$ intersects the surface S at the points P and Q.

So at P the cosine of the angle between the tangent vector and the normal vector is given by

$$\cos\theta_1 = \frac{\nabla F(1,1,2)\cdot \boldsymbol{r}'(1)}{|\nabla F(1,1,2)||\boldsymbol{r}'(1)|} = \frac{\begin{pmatrix}2\\2\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\\2\end{pmatrix}}{\sqrt{9}\sqrt{5}} = \frac{4}{3\sqrt{5}},$$

by the definition of the dot product. Thus the angle θ_1 is $\arccos\frac{4}{3\sqrt{5}}$. At Q we compute similarly for the cosine that

$$\cos\theta_2 = \frac{\begin{pmatrix}2\\2\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\\-6\end{pmatrix}}{3\sqrt{37}} = \frac{-4}{3\sqrt{37}}.$$

Notice that $\cos\theta_2$ is negative. This is because here the curve is going *into* the surface, while the normal is up (so the angle is $> \frac{\pi}{2}$). Usually we want the acute angle so we take

$$\theta_2 = \arccos\frac{4}{3\sqrt{37}}.$$

Here's a further example we didn't cover in class.

Example. Find the equation of the tangent line to the curve of intersection of the surfaces

$$f(x,y,z) = yz + zx + xy - 5 = 0,$$
$$g(x,y,z) = x^2 + y^2 + z^2 - 6 = 0,$$

at the point $(x,y,z) = (1,1,2)$.

It's a bit difficult to parametrise the curve of intersection directly. Instead we use the idea that the tangent line has to be normal to the normal vectors of both of the surfaces. We compute

$$\nabla f = \begin{pmatrix} y+z \\ z+x \\ x+y \end{pmatrix}, \qquad \nabla g = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix},$$

so that at $(1,1,2)$ we have

$$\nabla f(1,1,2) = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}, \nabla g(1,1,2) = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}.$$

A vector perpendicular to both of these is then necessarily parallel to their cross product (so it lies on the tangent planes of both of the surfaces):

$$\nabla f(1,1,2) \times \nabla g(1,1,2) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 3 & 2 \\ 2 & 2 & 4 \end{vmatrix} = \begin{pmatrix} 8 \\ -8 \\ 0 \end{pmatrix}.$$

We can factor out the constant without affecting the direction, so we have a vector $\left(\begin{smallmatrix} 1 \\ -1 \\ 0 \end{smallmatrix}\right)$ which points in the direction of the tangent line. By the vector equation of a line we can then write the tangent line as

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

where $\lambda \in \mathbb{R}$ is a parameter and $\boldsymbol{r} = \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)$.

4.7 Extremal Values

Recommended problems from §14.7: **31, 35, 40, 60**.

Suppose $f\colon [a,b] \to \mathbb{R}$ is function of one variable. If we want to find the maximum and minimum of f on $[a,b]$ then we know that we have to look at the interior critical points of f (so $c \in (a,b)$ such that $f'(c) = 0$) as well as the value of the function at the end points. On the other hand we know that if we look at a critical point then it's either a local minimum or a maximum or an inflection point. If we want to know more precisely then we have to look at the second derivative of f. From Figure 4.14 you can see how close to the local minima and maxima the function has to be larger or smaller, respectively, around the local extremum point.

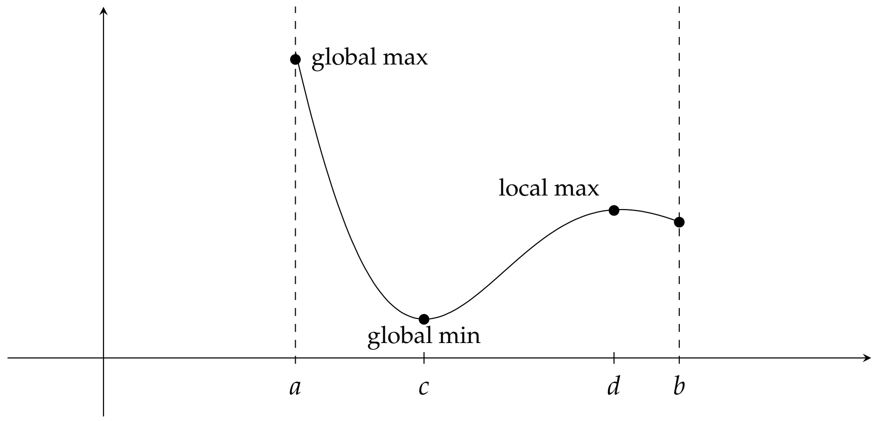


Figure 4.14: Extrema of $f(x) = y$.

Let's suppose now that we have a scalar function of several variables, say, $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}^3$. Then if f has a local minimum or maximum at $\boldsymbol{a} \in D$ and $\boldsymbol{a}$ is in the interior of D then we can see that

$$\nabla f(\boldsymbol{a}) = \mathbf{0}.$$

Any point at which the gradient vanishes is called a **critical point**. What about the converse? If a function has a critical point is it necessarily a local minimum or maximum? The answer is no, as demonstrated by the following example.

Example. Let

$$f(x, y) = x^2 - y^2.$$

Then the gradient of f is

$$\nabla f = \begin{pmatrix} 2x \\ -2y \end{pmatrix}.$$

We see that this has one critical point at the origin since $\nabla f(0, 0) = \mathbf{0}$. There $f(0, 0) = 0$. Is it a local min or max? Well, along the x axis the function is always positive for $x \neq 0$, so the origin cannot be a local maximum. On the other hand, along the y axis the value of the function is always negative for $y \neq 0$ so $(0, 0)$ cannot be a local minimum either. Such a point is called a **saddle point** since the neighbourhood around that point can sometimes resemble a saddle, as seen in Figure 4.15.

So how do we determine whether a critical point is a local min/max or a saddle point? In one variable the behaviour at a critical point is determined by the second derivative, which also is the second order term in the Taylor expansion of the function. There is no immediate analogue since now we have multiple second (partial) derivatives to consider (how many?). To see what happens we need to look at the multiple variable version of Taylor's theorem. For simplicity, let's work with $f(x, y)$.

Taylor's theorem in two variables says then that if f is smooth (and analytic[4]) at a

[4] In our context analytic just means that the function has a power series expansion. For functions of a complex variable differentiability is equivalent to analyticity!

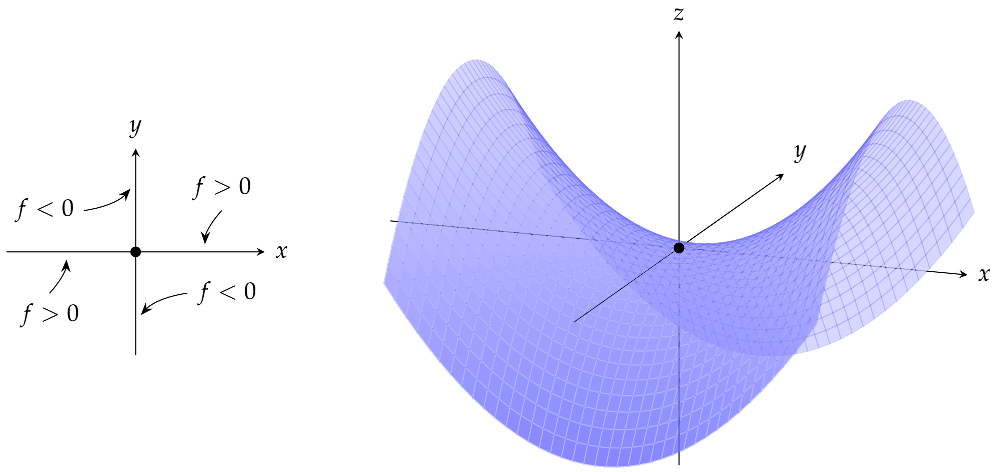


Figure 4.15: A saddle point of $f(x, y) = 0$ at the origin.

point $(a, b) \in \mathbb{R}^2$ then for (x, y) in some open disk centred at (a, b) we have

$$\begin{aligned} f(x,y) &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-a) \\ &\quad + \frac{1}{2!}\left(f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right) + \dots \end{aligned} \tag{4.15}$$

We only need the expansion up to second degree, but if you want to work out higher order terms then we can rewrite (4.15) in a more convenient form. You do not need to know how to do this, but I'll show you anyway. Let's define a new operator $\boldsymbol{D} = \left(\begin{smallmatrix} x-a \\ y-b \end{smallmatrix}\right) \cdot \nabla$. Then, for example,

$$\begin{aligned} \frac{1}{2!}\boldsymbol{D}^2 f(a,b) &= \frac{1}{2!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right)^2 f(a,b) \\ &= \frac{1}{2!}\left((x-a)^2\frac{\partial^2}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2}{\partial x \partial y} + (y-b)^2\frac{\partial^2}{\partial y^2}\right) f(a,b), \end{aligned}$$

which is exactly what we have for the second order term in (4.15). Similarly, the third order term in the Taylor expansion would then be

$$\begin{aligned} \frac{1}{3!}\boldsymbol{D}^3 f(a,b) &= \frac{1}{3!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right)^3 f(a,b) \\ &= \frac{1}{3!}\left(f_{xxx}(a,b)(x-a)^3 + 3f_{xxy}(a,b)(x-a)^2(y-b)\right. \\ &\quad \left. + 3f_{xyy}(a,b)(x-a)(y-b)^2 + f_{yyy}(a,b)(y-b)^3\right). \end{aligned}$$

Anyway, going back to (4.15), we already saw that the first order term can be written in terms of the gradient (when we found the tangent plane to a surface). In fact, we prefer to write the expansion as

$$\begin{aligned} f(x,y) = f(a,b) + \nabla f(a,b) \cdot \begin{pmatrix} x-a \\ y-b \end{pmatrix} \\ + \frac{1}{2} \begin{pmatrix} x-a & y-b \end{pmatrix} H \begin{pmatrix} x-a \\ y-b \end{pmatrix} + \dots, \end{aligned} \tag{4.16}$$

where

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(a,b) & \frac{\partial^2 f}{\partial x \partial y}(a,b) \\ \frac{\partial^2 f}{\partial x \partial y}(a,b) & \frac{\partial^2 f}{\partial y^2}(a,b) \end{pmatrix}$$

is called the **Hessian** matrix of f. It is a matrix that carries information about all the second partial derivatives of f at (a,b). By Clairaut's Theorem this is a symmetric matrix (so it's equal to its transpose). For a scalar function of n variables $f(x_1,\dots,x_n)$ the Hessian is analogously an $n \times n$ symmetric matrix given by

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ & \frac{\partial^2 f}{\partial x_2^2} & & \vdots \\ & & \ddots & \\ & & & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix},$$

where by leaving the lower half of the matrix blank we indicate that it is just the content in the upper half reflected along the diagonal (since H is symmetric). In other words we can write the Hessian matrix as $H = (a_{ij})_{i,j=1,\dots,n}$, where $a_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$.

We can now state the general second derivative test in terms of the Hessian matrix H.

General Second Derivative Test. *Suppose $f : D \to \mathbb{R}$ is a function with domain $D \subseteq \mathbb{R}^n$ and let $\boldsymbol{a} \in D$ be an* interior *point of D. Suppose that $\boldsymbol{a}$ is a critical point, so that $\nabla f(\boldsymbol{a}) = \mathbf{0}$, and all the second partial derivatives of f are continuous at $\boldsymbol{a}$. Let H be the Hessian of f at $\boldsymbol{a}$. Then,*

A) *if H is positive definite at $\boldsymbol{a}$, then f has a local minimum at $\boldsymbol{a}$,*

B) *if H is negative definite at $\boldsymbol{a}$, then f has a local maximum at $\boldsymbol{a}$,*

C) *if H is indefinite at $\boldsymbol{a}$, then f has a saddle point at $\boldsymbol{a}$,*

D) *otherwise this test gives no information and we have to investigate higher order terms in the Taylor expansion of f.*

Since H is a real symmetric $n \times n$ matrix, we can diagonalise it. This means that I can find an invertible matrix P and a diagonal matrix $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ such that

$$H = P^{-1}DP = P^{-1}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} P.$$

In particular, the diagonal entries of D, $\lambda_1, \ldots, \lambda_n$, are the *eigenvalues* of H. Thus we can restate the above conditions in terms of the eigenvalues of H:

A) $\iff$ all eigenvalues of H are positive,

B) $\iff$ all eigenvalues of H are negative,

C) $\iff$ H has both positive and negative eigenvalues (and possibly zeros for 3×3 matrices and larger).

In the case of 2 variables this can be simplified even further. Let $f(x, y)$ be a function of two variables with a critical point (a, b) and a Hessian H at (a, b). We can diagonalise H as

$$H = P^{-1}\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} P,$$

where P is an invertible matrix and λ_1 and λ_2 are the eigenvalues of H. Then, from your linear algebra course we know that the determinant and trace of H can be expressed in terms of the eigenvalues (we say that they are *invariants* of the matrix). So if we denote the matrix with only the eigenvalues on the diagonal by D, then $\det H = \det D$ and $\operatorname{tr} H = \operatorname{tr} D$. That is,

$$\begin{aligned} \det H &= \lambda_1 \lambda_2, \\ \operatorname{tr} H &= \lambda_1 + \lambda_2. \end{aligned}$$

If we want to determine whether we have a local min, max or a saddle point, we have to determine whether both eigenvalues are positive, negative or of opposite signs, respectively. We can now do this (for 2×2 case only) directly through the determinan and trace. Observe that

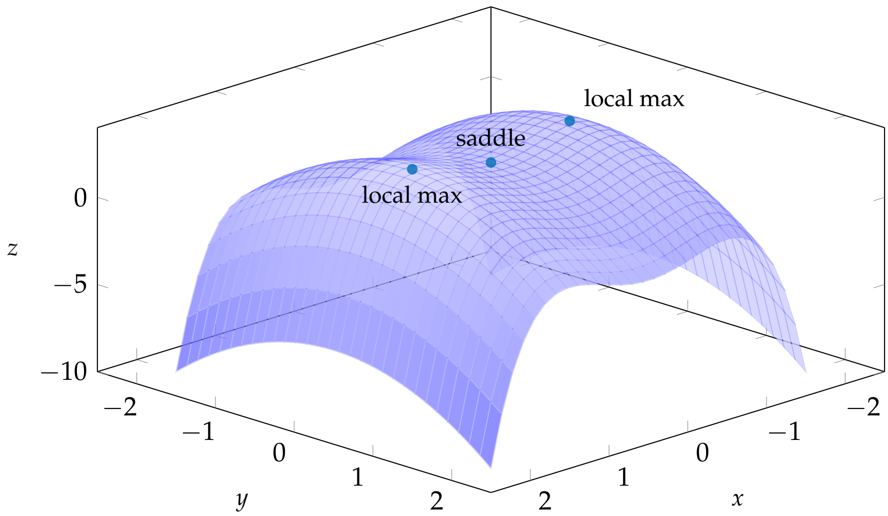


Figure 4.16: The critical points of $f(x,y) = 2 - x^4 + 2x^2 - y^2$.

- if $\det H < 0$ then $\lambda_1 \lambda_2 < 0$ so λ_1 and λ_2 necessarily have opposite signs. This means that we are in case **C)**, which gives a *saddle point*.
- if $\det H > 0$ then $\lambda_1 \lambda_2 > 0$ so all we can say is that both of the eigenvalues have the same sign, so either both are positive or both negative. To determine which case we are in we look at the trace:
  - if $\operatorname{tr} H > 0$ then $\lambda_1 + \lambda_2 > 0$ so in particular both $\lambda_1, \lambda_2 > 0$, which gives a *local minimum*.
  - if $\operatorname{tr} H < 0$ then $\lambda_1 + \lambda_2 < 0$ so similarly both $\lambda_1, \lambda_2 < 0$ and we have a *local maximum*.
- finally, if $\det H = 0$ then we are in case **D)** and the test is inconclusive.

We can summarise this in a convenient list as follows. Let H be the Hessian of $f(x,y)$ at (a,b). Then

1) $\det H < 0 \implies$ saddle point;
2) $\det H > 0$ and $\operatorname{tr} H > 0 \implies$ local MIN;
3) $\det H > 0$ and $\operatorname{tr} H < 0 \implies$ local MAX;
4) $\det H = 0 \implies$ test is inconclusive.

Let's do some examples.

Example. Let
$$f(x,y) = 2 - x^4 + 2x^2 - y^2.$$
We want to find and classify the critical points of f.

First, in order to find the critical points we need to compute the gradient.

$$\nabla f(x,y) = \begin{pmatrix} 4x - 4x^3 \\ -2y \end{pmatrix}.$$

If we set $\nabla f = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ then we get the two equations $x - x^3 = 0$ and $y = 0$ (after simplifying). The first one of these factorises as $x(1-x^2) = 0$, which means that either $x = 0$ or $x = 1$ or $x = -1$. Thus we get three critical points

$$(0,0),\ (1,0),\ \text{and}\ (-1,0).$$

To classify these we need to compute the Hessian H.

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 4 - 12x^2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Now we want to look at the determinant and trace of the Hessian at each of our critical points to determine local minima and maxima. It's also useful to get in to a habit of recording the value of the function at each critical point. This is important when you have e.g. multiple local maxima and you want to see which one is the largest.

At $(0,0)$: Here $f(0,0) = 2$, and the Hessian is

$$H = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}.$$

Thus $\det H = -8 < 0$ which means that we have a saddle point.

At $(1,0)$: We have $f(1,0) = 3$ and

$$H = \begin{pmatrix} -8 & 0 \\ 0 & -2 \end{pmatrix},$$

so that $\det H = 16 > 0$, which means we have either a local minimum or a local maximum. To determine which we look at the trace: $\operatorname{tr} H = -8 - 2 = -10 < 0$ so we have a local maximum.

At $(-1,0)$ We have as before $f(1,0) = 3$ and we also get the same Hessian

$$H = \begin{pmatrix} -8 & 0 \\ 0 & -2 \end{pmatrix}.$$

Hence $\det H > 0$ and $\operatorname{tr} H < 0$ so we again have a local maximum.

You can see the critical points of f in Figure 4.16.

Example. Now, let

$$f(x,y) = xye^{-(x^2+y^2)/2}.$$

Again, we first compute the gradient of $\nabla f(x,y)$.

$$\nabla f(x,y) = \begin{pmatrix} e^{-(x^2+y^2)/2}(y - x^2y) \\ e^{-(x^2+y^2)/2}(x - xy^2) \end{pmatrix}.$$

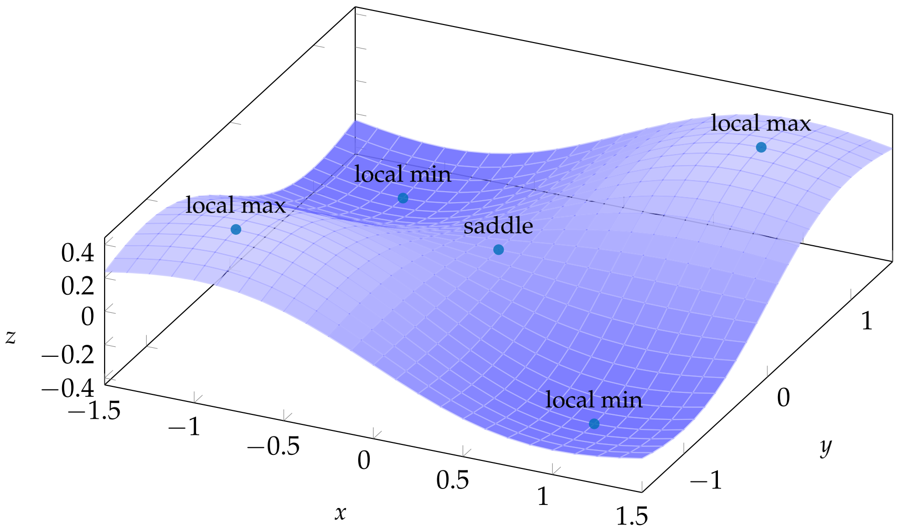


Figure 4.17: The critical points of $f(x,y) = xye^{-(x^2+y^2)/2}$.

If we set it equal to **0** we get the simultaneous equations $y(1-x^2) = 0$ and $x(1-y^2) = 0$. The first one has solutions $y = 0$ or $x = 1$ or $x = -1$, while the second one has $x = 0$ or $y = 1$ or $y = -1$. These conditions give us the following five points:

$$(0,0),\ (1,1),\ (1,-1),\ (-1,1),\ (-1,-1).$$

You can see these points in Figure 4.17. Next to compute the Hessian we first compute the second partial derivatives (we can use the first partial derivatives from the gradient for this)

$$\frac{\partial^2 f}{\partial x^2} = -xy(3-x^2)e^{-(x^2+y^2)},$$
$$\frac{\partial^2 f}{\partial y^2} = -xy(3-y^2)e^{-(x^2+y^2)},$$
$$\frac{\partial^2 f}{\partial x \partial y} = (1-y^2)(1-x^2)e^{-(x^2+y^2)/2}.$$

Now let's look at the Hessian at each of these points.

At (0, 0): Here $f(0,0) = 0$. We find the Hessian by substituting $x = 0$ and $y = 0$ into the above partial derivatives:

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $\det H = -1 < 0$ so f is a saddle point.

At (1, 1): We have $f(1,1) = e^{-1}$. Then

$$H = e^{-1}\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix},$$

and $\det H = 4e^{-2} > 0$. So we have either a local minimum or a local maximum. Since $\operatorname{tr} H = -4e^{-1} < 0$ it follows that $(1,1)$ is a local maximum. The same work applies to the point $(-1,-1)$.

At (1, −1): Now $f(1,-1) = -e^{-1}$ and

$$H = e^{-1}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then $\det H = 4e^{-2} > 0$ and $\operatorname{tr} H = 4e^{-1} > 0$ so we have a local minimum. We get the same result at $(-1,1)$.

Actually, we can say a bit more. If we look at the limit of $f(x,y)$ when the distance from the origin tends to ∞, that is $|\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)| \to \infty$ (so in polar coordinates $r \to \infty$), we can see that $f(x,y) \to 0$. This means that our function is in fact bounded and as it is continuous everywhere we can conclude that the local minima and maxima that we found are in fact global (absolute) minima and maxima.

This kind of trick does not always work (e.g. if the function is unbounded). Next time we'll see that for continuous functions on closed bounded domains we can always find a global maximum and a minimum.

Let's do a few more examples of finding extreme values. The first one is an example I did in class.

Example. We want to find the point on the plane

$$x - 2y + 3z = 6,$$

which is closest to the point $(0, 1, 1)$.

Before we can begin, we have to understand properly what the question is actually asking for. What is the function we are trying to minimise? It is the *distance* function of (x, y, z) from the point $(0, 1, 1)$, where (x, y, z) lies on the given plane. You should not work with the equation of the plane directly, it will not be helpful. The distance d between any point (x, y, z) in space and the point $(0, 1, 1)$ is of course given by

$$d = \sqrt{(x-0)^2 + (y-1)^2 + (z-1)^2}.$$

Since d is always non-negative, we can work with the simpler function of the square of the distance

$$d^2 = x^2 + (y-1)^2 + (z-1)^2.$$

This is still not quite what we want. We need to restrict ourselves only to points that lie on the plane. In this example this is easy, because for any y and z, I can find the corresponding x through the equation of the plane. So we substitute for x (or any of the other variables, it doesn't matter) in the distance function and get a function f of y and z given by

$$f(y, z) = (6 + 2y - 3z)^2 + (y-1)^2 + (z-1)^2.$$

Another important observation is that since the distance function is continuous (and positive) and when $\left|\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)\right| \to \infty$, the distance blows up to ∞. This means that necessarily there has to be a point on the plane with the smallest distance from $(0, 1, 1)$. So if we find one critical point then necessarily it gives the minimum. We compute that

$$\nabla f(y, z) = \begin{pmatrix} 10y + 22 - 12z \\ 20z - 38 - 12y \end{pmatrix}.$$

If we set $\nabla f(y, z) = \mathbf{0}$, then we get the system

$$\begin{cases} 10y + 22 - 12z = 0, \\ 20z - 38 - 12 = 0, \end{cases}$$

of two linear equations in two variables, so we can find a solution. Solving this system gives us the values

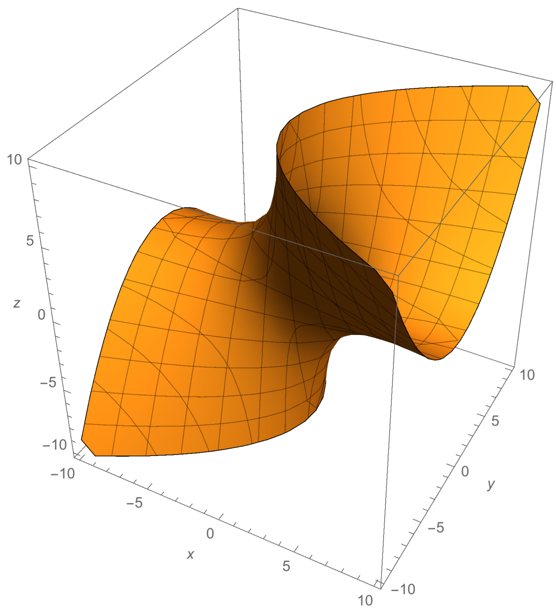
$$y = \frac{2}{7}, \text{ and } z = \frac{29}{14}.$$

We get the corresponding value of x through the equation of the plane so that

$$x = 6 + 2 \cdot \frac{2}{7} - 3 \cdot \frac{29}{14} = \frac{5}{14}.$$

Thus we have only one critical point $(\frac{5}{14}, \frac{2}{7}, \frac{29}{14})$, which is the closest point on the plane to the point $(0, 1, 1)$. This smallest distance is then given by

$$d = \sqrt{\left(\tfrac{5}{14}\right)^2 + \left(\tfrac{2}{7} - 1\right)^2 + \left(\tfrac{29}{14} - 1\right)^2} = \frac{5}{\sqrt{14}}.$$



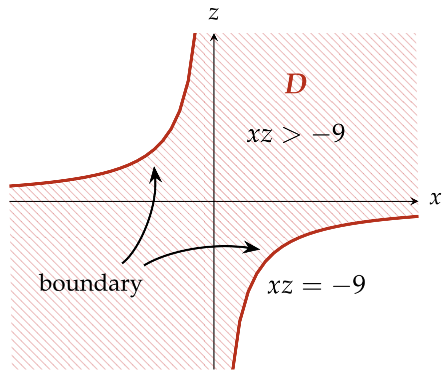


Figure 4.18: The surface $y^2 = 9 + xz$ and the domain of possible x and z values on the xz-plane (this is also what the surface looks like from the side).

The next example I didn't do in class, it is exercise 44 from section §14.7 in the book.

Example. Find the point(s) on the surface

$$y^2 = 9 + xz, \tag{4.17}$$

that are closest to the origin. In this case our (general) distance function is just

$$d^2 = x^2 + y^2 + z^2$$

since we are looking at the distance from the origin. We can now substitute for y^2 to get a function for the distance of points on the surface from the origin. We let

$$f(x, z) = x^2 + 9 + xz + z^2.$$

Again, by similar argument as before, there has to be a point with a minimal distance. However, we have to be a bit careful here. By the definition of the surface $9 + xz$ has to be non-negative (since it's equal to a square, y^2). Thus we get the restriction $9 + xz \geq 0$ or $xz \geq -9$. This condition defines a set D on the xz-plane, and we have to look at f separately in the interior and on its boundary (we'll see this again when we consider restricted domains), which you can see in Figure 4.18.

We first consider the interior of D and look for critical points of f. We get

$$\nabla f(x, z) = \begin{pmatrix} 2x + z \\ x + 2z \end{pmatrix}.$$

If we set $\nabla f(x,z) = \mathbf{0}$ we get the equations $2x + z = 0$ and $x + 2z = 0$. These have the only solution $x = 0$ and $z = 0$. So any critical point has these x and z coordinates. We use (4.17) to find that the possible values of y are ± 3. So we get the points $(0,3,0)$ and $(0,-3,0)$. The distance to the origin from both of these points is 3.

Now we have to investigate the boundary when $y = 0$, i.e. $zx = -9$. So we are looking at points on our surface with $y = 0$, and $z = -9/x$, which gives a curve. We can parametrise this curve by setting $x = t$ and considering $t \in \mathbb{R} \setminus \{0\}$, so that we get $\boldsymbol{r}(t) = (t, 0, -9t^{-1})$. Define $g(t)$ to be the distance (squared) from the origin to any point on this curve:

$$g(t) = t^2 + 81t^{-2}.$$

This function is differentiable in its domain so we look for its critical points:

$$g'(t) = 2t - 162t^{-3} = 0.$$

Since $t \neq 0$ this is equivalent to $t^4 - 81 = 0$, which has solutions $t = \pm 3$. So we get two other candidates for the minimal distance by looking at the points $(3, 0, -3)$ and $(-3, 0, 3)$. The distance to the origin from both of these points is $3\sqrt{2}$ which is larger than the distance we found above, so we conclude that the smallest distance from our surface to the origin is 3 and it is attained at the two points $(0, \pm 3, 0)$.

Extreme Values on Restricted Domains

In this section we'll see more carefully how to deal with optimising functions when we restrict them to some given set (just like in the previous two examples we restricted the domain of the distance function to a subset of the space $\mathbb{R}^3$). We will consider the special case when the restriction defines a *closed* and *bounded* set. The crucial theorem is the following (you should already be familiar with the one variable version of this theorem).

Theorem 4.9. *Let D be a closed and bounded region $D \subset \mathbb{R}^n$ and suppose $f : D \to \mathbb{R}$ is a continuous function on D. Then f attains its absolute*[5] *minimum and maximum on D.*

In fact there is no need to even restrict to scalar functions in this theorem as it is true for functions from $\mathbb{R}^n$ to $\mathbb{R}^m$, but we'll only be using it in the above context.

In one variable we typically have a function $f : [a,b] \to \mathbb{R}$. The domain here is a closed and bounded set, so by the above theorem f has to attain its absolute minimum and maximum value somewhere in the closed interval $[a,b]$. Thus we know that we have to check

- interior critical points, i.e. $x \in (a,b)$ such that $f'(x) = 0$,
- the end points $f(a)$ and $f(b)$,
- any points in the interior where f is not differentiable.

Then we are guaranteed to find the maximal and minimal values of f and the points at which they occur (there could be several, even infinitely many). Now let's suppose we have a function $f : [0, \infty) \to \mathbb{R}$ given by $f(x) = \arctan x$. We know $f(x) \to \frac{\pi}{2}$ as

[5] i.e. global

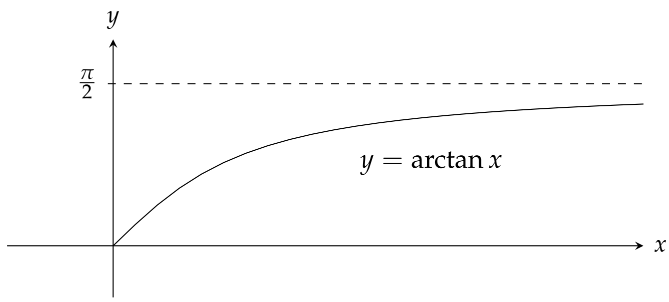


Figure 4.19: The graph of the function $f(x) = \arctan x$.

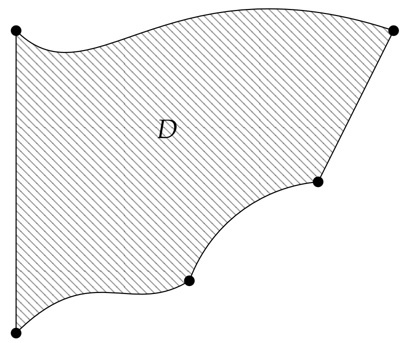


Figure 4.20: A domain D and its boundary with sides and some corners in $\mathbb{R}^2$.

$x \to \infty$ and that it is an increasing function, thus the graph of f looks like in Figure 4.19. We see that f takes its minimum value 0 at the point $x = 0$ in the interval. However, it never attains the maximum value $\pi/2$. Why does this not contradict Theorem 4.9? Well, we notice that the domain $[0, \infty)$ is not bounded (it is certainly closed, because its complement, $(-\infty, 0)$ is of course an open set), so the conditions of the theorem are not satisfied.

Let's move on to the higher variable case. Now the domain can be a more complex shape, like in Figure 4.20, for example. To find the absolute extremal values of a function $f(x, y)$ on such a domain we then have to check the following:

1) critical points of f in the interior of D (so points where $\nabla f = \mathbf{0}$),

2) values of f on the sides of D,

3) values of f at the corners of D,

4) possible points in the interior of D where f is not differentiable.

There are a couple of points to make here. First, when finding the critical points of f, it's important to be careful that the critical points actually lie in the interior of D and not outside or on the boundary of D. Second, if we are only interested in finding the extremal *values* of f (as opposed to e.g. also classifying the critical points), then it's not necessary to compute the Hessian matrix! This is because Theorem 4.9 guarantees that we can find points with the absolute minimal and maximal values of f, so we can just

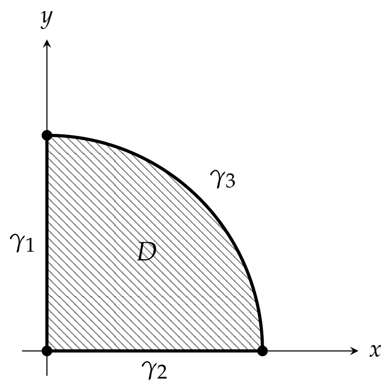


Figure 4.21: The domain D with sides γ_1, γ_2 and γ_3 and the three corner points.

compute the value of f at each of the points we get from the cases **1**) $-$ **4**) and compare the values to find the smallest and largest. Let's do an example.

Example. Consider

$$f(x,y) = xy^2,$$

in the region

$$D = \{(x,y) : x \geq 0, y \geq 0, x^2 + y^2 \leq 3\}.$$

First of all we notice D is a bounded region (because we can find a disk that contains all of D) and it is also closed (roughly speaking because none of the inequalities are strict). Thus Theorem 4.9 applies and we can find points either in the interior of D, on the sides of D or at the corners of D, where f attains its maximal and minimal values. Now, it's often helpful to sketch the domain to make sure you're not missing anything. We do this in Figure 4.21. We see the domain D has three sides γ_1, γ_2 and γ_3, as well as three corners at $(0,0)$, $(\sqrt{3},0)$ and $(0,\sqrt{3})$. Notice that f is differentiable everywhere. We start by finding the interior critical points.

1) For the interior critical points we get

$$\nabla f(x,y) = \begin{pmatrix} y^2 \\ 2xy \end{pmatrix},$$

and we set $\nabla f = \mathbf{0}$. This immediately tells us that for all the critical points $y = 0$. Thus none of them can be in the interior of D (since all points in the interior have $y > 0$). So we have no critical points in the interior and we get no possible points from here.

2) Next we look at the sides. The sides are curves, and we want to parametrise them and substitute to the definition of f. This gives us a new function of one variable and we'll then find its critical points.

$\boldsymbol{\gamma_1}$**:** For points on this side we have $(x,y) = (0,t)$ with $0 < t < \sqrt{3}$ and we get the function

$$g_1(t) = f(0,t) = 0.$$

Thus the value of f on the y-axis (and on γ_1) is 0.

γ_2: Now, $(x,y) = (t,0)$ and $0 < t < \sqrt{3}$ again. Thus

$$g_2(t) = f(t,0) = 0,$$

and we see as before that the value of f on γ_2 is 0.

γ_3: Now we are on the circular arc, so we parametrise by setting $(x,y) = (\sqrt{3}\cos t, \sqrt{3}\sin t)$ with $0 < t < \frac{\pi}{2}$. Substituting into the definition of f gives

$$g_3(t) = 3\sqrt{3}\cos t \sin^2 t.$$

This is a bit more interesting case than before. To find the critical points of g_3, we try to find points when the derivative is zero:

$$g_3'(t) = -3\sqrt{3}\sin^3 t + 6\sqrt{3}\cos^2 t \sin t = 0.$$

This simplifies to

$$2\cos^2 t - \sin^2 t = 0,$$

since $\sin t$ is non-zero for $t \in (0, \frac{\pi}{2})$. We use the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$ to simplify this to

$$3\cos^2 t = 1,$$

which gives

$$\cos t = \frac{1}{\sqrt{3}}.$$

Since our parametrisation was in terms of $\cos t$ and $\sin t$, we don't need to find the actual angle here, we just need to figure out what is the value of $\sin t$. To do this we use the same trigonometric identity as before to get that

$$\sin^2 t = 1 - \cos^2 t = 1 - \frac{1}{3} = \frac{2}{3}.$$

Since $\sin t$ has to be positive for $t \in (0, \frac{\pi}{2})$ we conclude that

$$\sin t = \sqrt{\frac{2}{3}}.$$

Thus we get one critical point of g_3 along γ_3 at $(x,y) = (1, \sqrt{2})$, where f takes the value $f(1,\sqrt{2}) = 2$.

3) Finally we check the value of f at the corners:

$$\begin{aligned} f(0,0) &= 0, \\ f(\sqrt{3},0) &= 0, \\ f(0,\sqrt{3}) &= 0. \end{aligned}$$

Now, combining the information from all of the cases we see that f has an absolute maximum value 2, which it attains at the point $(1, \sqrt{2})$. Also, the absolute minimum value of f on D is 0, which is attained along the part of the boundary of D on the x and y axes.

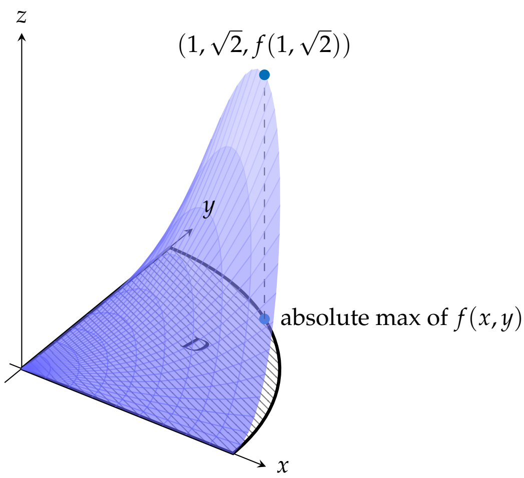


Figure 4.22: The highest and lowest points of the surface $z = f(x, y)$, with $f(x, y) = xy^2$, lying above the domain D.

Notice that while the whole discussion here is not about the surface $z = f(x, y)$, the extremal values of f we obtained still have a geometric meaning in terms of this surface. Basically what we are saying is that if we restrict the surface by only considering points $(x, y) \in D$, then as long as D is closed and bounded and f is continuous, then the surface $z = f(x, y)$ has a highest and lowest point(s) when restricted to D. In Figure 4.22 we see how this works in the case of our previous example.

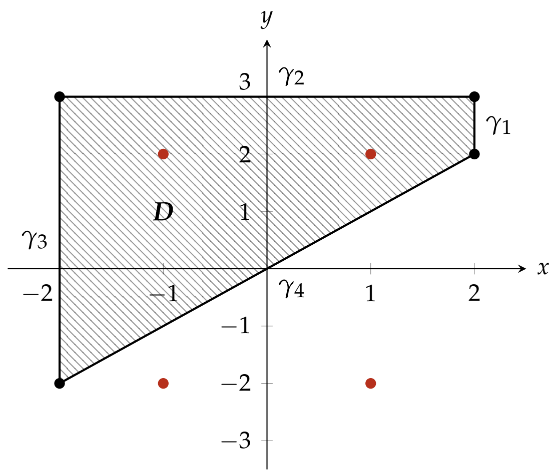


Figure 4.23: The quadrilateral D with sides $\gamma_1, \dots, \gamma_4$. Also pictured are the critical points of $f(x, y) = x^3 - 3x - y^3 + 12y$ in red.

Let's do one more example of extreme values on restricted domains before moving on.

Example. Let

$$f(x, y) = x^3 - 3x - y^3 + 12y,$$

and let D be the quadrilateral with corners $(-2, 3)$, $(-2, -2)$, $(2, 2)$ and $(2, 3)$. Denote the sides of D by $\gamma_1, \dots, \gamma_4$ as shown in Figure 4.23. We then proceed as before.

1) The gradient of f is

$$\nabla f = \begin{pmatrix} 3x^2 - 3 \\ -3y^2 + 12 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives us the equations

$$\begin{cases} x^2 - 1 = 0, \\ y^2 - 4 = 0, \end{cases}$$

which have the solutions $x = \pm 1$ and $y = \pm 2$. This gives us four points (as you see in Figure 4.23). But only two of them are in the interior of D, namely, $(1, 2)$ and $(-1, 2)$. The value of f at these points is

$$f(1, 2) = 14, \qquad f(-1, 2) = 18.$$

2) Now we look at the boundary.

$\boldsymbol{\gamma_1}$**:** We parametrise this as $(x, y) = (2, t)$, where $2 < t < 3$. Then we get

$$g_1(t) = f(2, t) = 2 - t^3 + 12t.$$

We look for critical points:

$$g_1'(t) = -3t^2 + 12 = 0,$$

which has solutions $t = \pm 2$. But neither of these lies in the interval $2 < t < 3$, so they are not in the interior of this side. Thus we get nothing from γ_1.

γ_2: Now $(x, y) = (t, 3)$ and $-2 < t < 2$. Define

$$g_2(t) = f(t, 3) = t^3 - 3t + 9.$$

Then

$$g_2'(t) = 3t^2 - 3 = 0.$$

We have the solutions $t = \pm 1$, which both give us points in the interior of $t \in (-2, 2)$. These values of t amount to the points $(1, 3)$ and $(-1, 3)$ and we compute

$$f(1, 3) = g_2(1) = 7,$$
$$f(-1, 3) = g_1(-1) = 11.$$

γ_3: Next up we got $(x, y) = (-2, t)$ and $-2 < t < 3$. Thus

$$g_3(t) = f(-2, t) = -2 - t^3 + 12t,$$

and

$$g_3'(t) = -3t^2 + 12 = 0.$$

So again we get $t = \pm 2$. This time, out of these two values only $t = 2$ is inside the interval $(-2, 3)$. So we get one possible candidate $(-2, 2)$ with the value of f there being

$$f(-2, 2) = g_3(2) = 14.$$

γ_4: Finally we have the diagonal side, which we parametrise as $(x, y) = (t, t)$, where $-2 < t < 2$. Then

$$g_4(t) = f(t, t) = 9t$$

and we see immediately that $g_4'(t) = 9$ which is never equal to 0 so we have no critical points of g_4 along this side.

3) To finish up, we check the corners:

$$f(2, 3) = 11, \qquad f(-2, 3) = 7,$$
$$f(-2, -2) = -18, \qquad f(2, 2) = 18.$$

We go over every single point we found and record the largest and smallest values that we found. This leads to the following observations:

- f has an (constrained) absolute maximum value 18 which it attains at the points $(-1, 2)$ and $(2, 2)$. Out of these the former is a critical point of f.

- The (constrained) absolute minimum value of f is -18, which is attained at $(-2, -2)$.

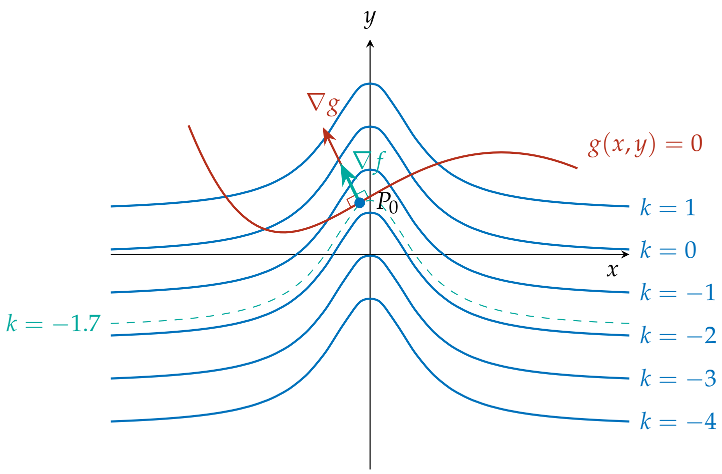


Figure 4.24: The level curves $f(x,y) = k$ and the curve $g(x,y) = 0$, with the level curve giving a constrained maximal value of f highlighted.

4.8 Lagrange Multipliers

Recommended problems from §14.7: **31, 35, 40, 50**.

The aim of this section is to introduce a better (well, to be fair this is subjective and depends on the problem, but often this might be easier) way to solve the kind of problems where we want to optimise (maximise or minimise) a function but we subject it to certain conditions. Think back to the examples with the distance function or the examples with restricted domains. So the situation is that we have a function f and we want to find the *constrained* extrema of this function subject to the constraint conditions $g_1 = 0, g_2 = 0, \ldots, g_k = 0$. In this course we'll mostly look at the case of one or two constraints (although the methods generalise easily). We'll also usually be working with functions of two or three variables.

The idea (originally by the Italian mathematician Joseph-Louis Lagrange) is to introduce a new variable. This transforms the problem into a higher dimensional one, but hopefully to something that's easier to solve. Let's see how he derived his method from scratch. We start with the easier case of a function $f(x,y)$ of two variables which we want to maximise subject to the constraint $g(x,y) = 0$. The function f takes values everywhere on the xy-plane, so we can represent it by its level curves, whereas $g(x,y) = 0$ represents a curve on that plane. You can see this in Figure 4.24. Consequently, we are looking for the maximal value of the function f as we traverse the curve $\mathscr{C}$ given by $g(x,y) = 0$. It turns out that if such a point exists it has to occur when the gradients of f and g are parallel. Why? Let's prove it by contradiction. Suppose P_0 is the point at which f has the largest value along $\mathscr{C}$, but that the gradients ∇f and ∇g are *not* parallel. This means that ∇f has a non-zero vector projection onto the tangent line of $\mathscr{C}$ at P_0 (since ∇g is perpendicular to the tangent line). For this to make sense we need to assume that $\nabla g \neq \mathbf{0}$. You can see this in Figure 4.25 where we have zoomed in at the point P_0 from the previous figure. It follows that the directional derivative of f in

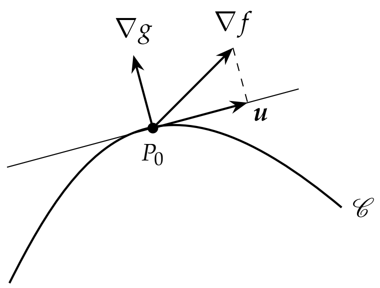


Figure 4.25: Vector projection $\boldsymbol{u}$ of ∇f to the tangent line of $\mathscr{C}$ at P_0.

the direction of the vector projection is positive so that f is increasing if we move along $\mathscr{C}$ in this direction. But this is a contradiction since we assumed that the value of f at P_0 was maximal. Thus ∇f and ∇g are parallel[6]. Recall that if two vectors are parallel, it means that they are scalar multiples of each other. Hence we can write that

$$\nabla f = \lambda \nabla g,$$

where $\lambda \in \mathbb{R}$ is called the **Lagrange multiplier**. Clearly this discussion only makes sense if such a point P_0 exists in the first place and if ∇f and ∇g both exist. Hence, we can summarise the method of Lagrange multiplier with one constraint as follows.

Lagrange Method for 1 Constraint. *Let $f(x,y)$ be a function subject to the constraint $g(x,y) = 0$. Define the **Lagrangian function**, $L(x,y,\lambda)$, as*

$$L(x,y,\lambda) = f(x,y) - \lambda g(x,y).$$

*Then any maximal (or minimal) value of f along the constraint $g(x,y) = 0$, if it exists and $\nabla g \neq \mathbf{0}$, (called a **constrained extremum**) necessarily satisfies the condition $\nabla L = \mathbf{0}$. In other words*

$$\frac{\partial L}{\partial x} = 0, \qquad \frac{\partial L}{\partial y} = 0, \qquad \frac{\partial L}{\partial \lambda} = 0.$$

Let's make three remarks. First, in the above we see that we get 3 equations in 3 variables, so it should be possible for us to find a solution. Second, this theorem generalises in the obvious way to the higher variable case by constructing the function $L(x,y,z,\lambda)$ etc. In a bit we'll also see how to generalise to two constraints. Finally, it's important to notice that this theorem does not guarantee us that *any* point we find is a constrained extremum. It only tells us that any constrained extremum has to satisfy these conditions. So the theorem gives us a necessary, but not sufficient condition. Usually when solving the equations in Theorem 4.8, we end up with a bunch of points. We then have to check manually each of those points to determine where the constrained maximal and minimal values occur. Remember that if the constraint defines a closed and bounded set, and if f is continuous on it, then these constrained maximal and minimal values are guaranteed to exist (why?). Putting all this together gives us the following steps to optimise the function $f(x,y)$ subject to $g(x,y) = 0$. We have to look for

[6] This is our first serious example of a method of proof called *proof by contradiction*. It is a very powerful tool.

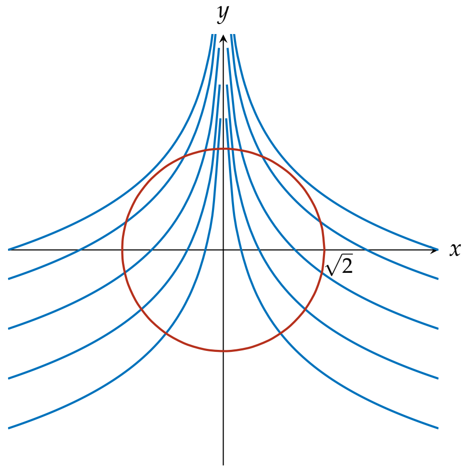


Figure 4.26: Level curves of the function $f(x,y) = xe^y$ and the constraint $x^2 + y^2 = 2$.

1) critical points of L, so points where $\nabla L = \mathbf{0}$,

2) points where $\nabla g = \mathbf{0}$ on the constraint set,

3) points where ∇f or ∇g don't exist on the constraint set,

4) any corners/endpoints of the constraint set (since they don't have a well-defined tangent line).

Before looking at the two constraint version, let's do an example. It's also a good idea to try to work through some of the earlier constrained examples, but now by using the method of Lagrange multipliers and to verify that you arrive at the same result.

Example. Find the maximum and minimum values of

$$f(x,y) = xe^y,$$

subject to the constraint

$$x^2 + y^2 = 2.$$

So the question is asking us to optimise the function f on the circle of radius $\sqrt{2}$ centred at the origin, see Figure 4.26. Notice that the set defined by the constraint is closed and bounded, and that f is continuous on it. It follows that f necessarily has an absolute minimum and maximum value on the circle. To find these values we want to construct the Lagrangian function. First, set $g(x,y) = x^2 + y^2 - 2$. Then the Lagrangian function L is given by

$$\begin{aligned} L(x,y,\lambda) &= f(x,y) - \lambda g(x,y) \\ &= xe^y - \lambda(x^2 + y^2 - 2). \end{aligned}$$

Now we look for points such that $\nabla L = \mathbf{0}$. This gives the conditions

1: $\frac{\partial L}{\partial x} = e^y - 2\lambda x = 0$,

2: $\frac{\partial L}{\partial y} = xe^y - 2\lambda y = 0$,

3: $\frac{\partial L}{\partial \lambda} = x^2 + y^2 - 2 = 0.$

Observe from the first equation that $\lambda \neq 0$ because otherwise we would get $e^y = 0$, which is impossible. Then from $\mathbf{1} \cdot x - \mathbf{2}$ we get

$$-2\lambda x^2 + 2\lambda y = 0.$$

Since $\lambda \neq 0$, we can divide to get

$$y = x^2.$$

Notice that this implies that $y \geq 0$. We substitute the above relation back to **3** which gives an equation for y,

$$y + y^2 - 2 = 0 \quad \iff \quad (y-1)(y+2) = 0.$$

This gives two possible solutions $y = 1$ and $y = -2$. But if $y = -2$ then **3** becomes

$$x^2 = -2,$$

which is impossible, so we discard this y value. Thus $y = 1$ and we get the possible x values from **3**:

$$x^2 + 1 - 2 = 0 \quad \iff \quad x^2 = 1 \quad \iff \quad x = \pm 1.$$

Thus we get two candidates for the extremal values $(1, 1)$ and $(-1, 1)$. These points correspond to the multiplier values $\lambda = \pm e/2$, respectively. Although in this question we don't even need to find these! At the candidate points we have

$$f(1, 1) = e, \qquad\qquad f(-1, 1) = -e.$$

By our earlier remark f has to attain an absolute minimum and maximum somewhere. Since $\nabla g \neq \mathbf{0}$ on the circle and ∇f and ∇g are well-defined everywhere, and the constraint set has no corners, we conclude that the two candidate points we found have to give us the constrained maximum and minimum value of f. We can also phrase this question in terms of the surface corresponding to the graph of f, given by $f(x, y) = z$. Then the constraint gives us a path of possible x and y coordinates on the surface which we can walk along. And we're then trying to find the highest and lowest point along this path. This is depicted in Figure 4.27.

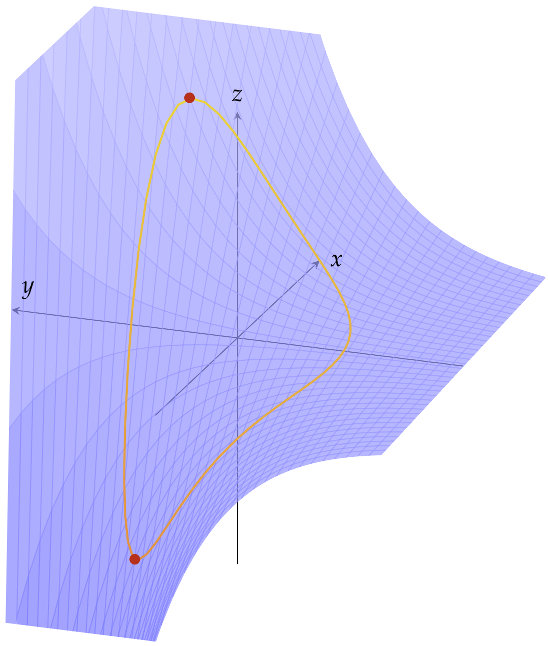


Figure 4.27: The surface $z = f(x, y)$ and the path on it given by the constraint $g(x, y) = 0$. The highest and lowest (i.e. the constrained maximum and minimum of f) points on the surface are highlighted.

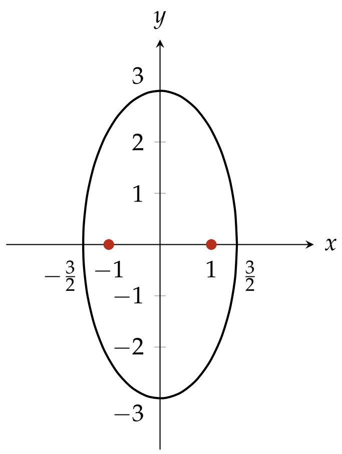


Figure 4.28: The constraint $4x^2 + y^2 \leq 9$ defines an elliptical disk with its boundary included. Depicted also the interior critical points of $f(x, y) = x^3 - 3x - y^2$.

We can use Lagrange's method to help us solve the problem of maximising a function on restricted domains. We do this by looking for critical points in the interior as before and then applying Lagrange on the boundary.

Example. Find the minimum and maximum values of

$$f(x, y) = x^3 - 3x - y^2$$

in the set defined by the condition

$$4x^2 + y^2 \leq 9.$$

Notice that the boundary condition defines a closed and bounded subset of $\mathbb{R}^2$. Since f is continuous everywhere, it will necessary attain a minimal and maximal value somewhere on the set. We treat the interior and the boundary of the constraint set separately. In the interior we just look for critical points of f:

$$\nabla f = \begin{pmatrix} 3x^2 - 3 \\ -2y \end{pmatrix}.$$

If $\nabla f = \mathbf{0}$, then it follows immediately that $y = 0$ and $x^2 = 1$ so $x = \pm 1$. Hence we have two interior critical points $(1, 0)$ and $(-1, 0)$, which are shown in Figure 4.28. On these, f has the following values

$$f(1, 0) = -2, \qquad\qquad f(-1, 0) = 2.$$

Next we look at the boundary of the constraint set. Thus we restrict x and y values to be on the ellipse $4x^2 + y^2 = 9$. We apply Lagrange's method to optimise f subject to this constraint. The Lagrangian is

$$L(x, y, \lambda) = x^3 - 3x - y^2 - \lambda(4x^2 + y^2 - 9).$$

We set the gradient of L to be equal to $\mathbf{0}$, which gives the equations

$$\begin{aligned} 3x^2 - 3 - 8\lambda x &= 0, \\ -2y - 2\lambda y &= 0, \\ 4x^2 + y^2 - 9 &= 0. \end{aligned}$$

Usually it's a good strategy to try to factorise any equations that you can (sometimes it's not possible). For example, from the second equation we immediately get that

$$2y(1+\lambda) = 0, \tag{4.18}$$

which tells us that either $y = 0$ or $\lambda = -1$. We treat the two cases separately. First, suppose that $\lambda = -1$. Then from the first equation we eliminate λ to get a quadratic in x:

$$3x^2 + 8x - 3 = 0.$$

This can easily be solved to get two possible values for x, namely, $x = \frac{1}{3}$ or $x = -3$ (check yourself!). We can use the constraint equation (the third equation above) to try to find the corresponding y values. For $x = -3$ we get $y^2 = 9 - 4 \cdot 9$, which is impossible since $y^2 \geq 0$. Thus we discard this x-value. Then, for $x = \frac{1}{3}$ we get that $y^2 = \frac{77}{9}$, so that we get the following two points:

$$(\tfrac{1}{3}, \tfrac{\sqrt{77}}{3}) \text{ and } (\tfrac{1}{3}, -\tfrac{\sqrt{77}}{3}).$$

In the second case (from (4.18)) we have $y = 0$. Then from the third equation in the gradient of L, we immediately get $x^2 = \frac{9}{4}$, which gives the two points

$$(\tfrac{3}{2}, 0) \text{ and } (-\tfrac{3}{2}, 0).$$

To find the constrained minimal and maximal values it remains to evaluate f at each of the points. We find that

$$f(\tfrac{3}{2}, 0) = -\frac{9}{8}, \qquad f(\tfrac{1}{3}, \tfrac{\sqrt{77}}{3}) = -\frac{257}{27},$$
$$f(-\tfrac{3}{2}, 0) = \frac{9}{8}, \qquad f(\tfrac{1}{3}, -\tfrac{\sqrt{77}}{3}) = -\frac{257}{27}.$$

We also have to remember to include the two interior critical points:

$$f(1,0) = -2, \qquad f(-1,0) = 2.$$

Comparing these values we see that f has an absolute (constrained) maximum value 2, which it attains at the point $(-1, 0)$, whereas the absolute (constrained) minimum value of f is $-\frac{257}{27}$, which is attained at the two points $(\frac{1}{3}, \pm\frac{\sqrt{77}}{3})$. The geometric situation corresponds to considering the height of the surface $z = f$ in a region which, when projected to the xy-plane, is the elliptical disk $4x^2 + y^2 \leq 9$.

Like I mentioned in the lectures, it is also possible to complete this problem in one big swoop by using a modified version of the Lagrangian function. This is *not* part of the course (so you don't have to know it). Anyway, if we introduce an extra variable u and define

$$L(x, y, z, \lambda, u) = f(x, y) - \lambda(g(x, y) + u^2),$$

then if you look for points when the gradient of L is $\mathbf{0}$ you get 5 equations in 5 variables, and you'll find that you get exactly the 6 points we obtained in the previous example. Why does this work? Well, let's look at the partial derivative of L with respect to u being equal to zero. This gives

$$\frac{\partial L}{\partial u} = -2\lambda u = 0.$$

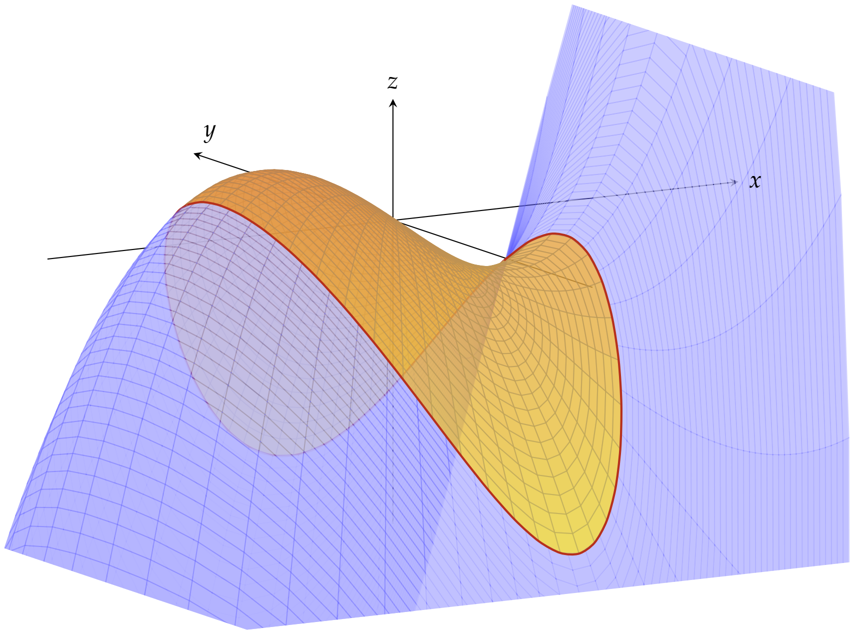


Figure 4.29: The graph of the function $f(x,y) = x^3 - 3x - y^2$ restricted to the set $4x^2 + y^2 \leq 9$ (in orange).

So either $u = 0$ or $u \neq 0$ (and then necessarily $\lambda = 0$). But if $u \neq 0$, then the partial derivative of L w.r.t. λ gives that

$$g(x,y) = -u^2.$$

Since u is non-zero this tells us that $g(x,y) \leq 0$, i.e. ($4x^2 + y^2 \leq 9$) so we're in the case of interior points of the domain! Then since λ is zero, the partial derivatives of L w.r.t. x and y just reduce to the equations we had when we set the gradient of f equal to zero. On the other hand, if $u = 0$, then the partial derivative of L w.r.t. λ just tells us that we are on the boundary of the constraint set (i.e. when the inequality becomes an equality). This is pretty neat! However neat it may be, for such a simple problem it's a bit like using a sledgehammer to crack a nut, as it's easier to just check the interior critical points separately. For more complex problems with many constraints given by inequalities, this ends up being a very useful method (for computers). The generalisation of the above trick to multiple inequalities gives us something called the *Kuhn–Tucker conditions* and it is a method used in *non-linear programming*.

Let's now look at the case of two constraints. First a quick remark. If we're in the case of functions of two variables, $f(x,y)$, and we have two constraints, say, $g_1(x,y) = 0$ and $g_2(x,y) = 0$. Then both of these constraints define some curve in the xy-plane. But we are considering f subject to *both* of the constraints at the same time, so this (usually) just leaves a finite (or countable) set of points for us to consider. Then it's just easier to evaluate f directly at each of these points, and compare the values. You can see this situation in Figure 4.30. Thus to have a more meaningful problem we consider functions of three variables, $f(x,y,z)$, subject to two constraints $g_1(x,y,z) = 0$ and $g_2(x,y,z) = 0$. The two constraints both define surfaces in $\mathbb{R}^3$, call them S_1 and S_2. Since we want to restrict f to both of these constraints, we need to look at the intersection of S_1 and S_2.

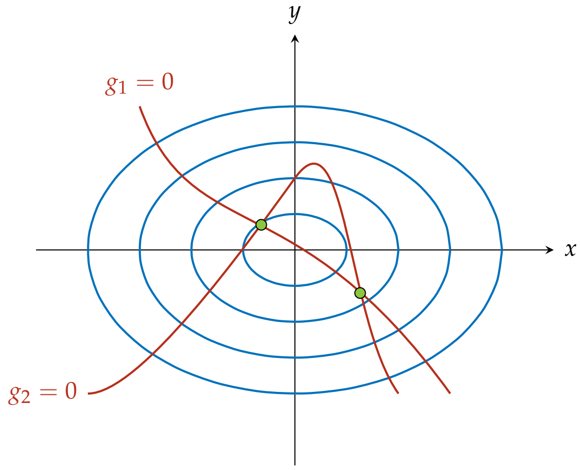


Figure 4.30: Two constraints in 2 variables usually just results in some set of points.

This defines a curve $\mathscr{C}$, as you can see in Figure 4.31. By a similar argument as in the

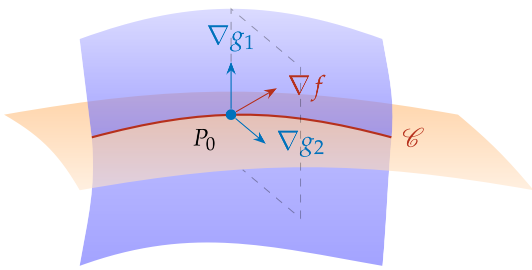


Figure 4.31: ∇f has to lie on the plane spanned by ∇g_1 and ∇g_2.

case of one constraint, if f has a maximal value at P_0 along the curve $\mathscr{C}$ then ∇f has to be perpendicular to this curve (otherwise it has a non-zero directional derivative tangent to the curve). Therefore ∇f lies on the plane spanned by ∇g_1 and ∇g_2 (here we have to assume that the gradients of the constraints are not parallel). This means that ∇f can be written as a linear combination of ∇g_1 and ∇g_2, that is, we have

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2,$$

where $\lambda, \mu \in \mathbb{R}$ are now the Lagrange multipliers. The corresponding Lagrangian function will then be a function of 5 variables given by

$$L(x, y, z, \lambda, \mu) = f(x, y, z) - \lambda g_1(x, y, z) - \mu g_2(x, y, z).$$

We again look for critical points of L, that is, $\nabla L = \mathbf{0}$. Now the equations can just be slightly harder to solve, but otherwise we proceed as before. This is best demonstrated through an example.

Example. Let

$$f(x, y, z) = x^2 + y^2 + z^2.$$

We want to optimise f subject to the constraints

$$x - y = 1, \qquad y^2 - z^2 = 1.$$

Notice that f actually represents the square of the distance of a point (x, y, z) from the origin. Therefore, geometrically, we're trying to find the minimal and maximal distance from the origin along the intersection of the two surfaces given by the constraint, this is depicted in Figure 4.32. It's possible to show that the set defined by the intersection of the surfaces gives an unbounded curve (or rather a disjoint union of two unbounded curves), so therefore there is no maximal distance. Moreover, since the distance function is continuous and positive, this means that there has to be a minimal distance. The Lagrangian is

$$L(x, y, z, \lambda, \mu) = x^2 + y^2 + z^2 - \lambda(x - y - 1) - \mu(y^2 - z^2 - 1).$$

The critical points of L satisfy the following equations

1) $2x - \lambda = 0,$

2) $2y + \lambda - 2\mu y = 0,$

3) $2z + 2\mu z = 0,$

4) $x - y - 1 = 0,$

5) $y^2 - z^2 - 1 = 0.$

To begin with, we notice that equation **1)** allows us to eliminate the multiplier λ immediately. We substitute this to equation **2)** to get

$$2y(1 - \mu) + 2x = 0, \tag{4.19}$$

which will be useful later. Now, we notice that equation **3)** can be factorised as

$$2z(1 + \mu) = 0.$$

This means that either $z = 0$ or $\mu = -1$. In the first case we obtain from **5)** immediately that $y^2 = 1$ so $y = \pm 1$. By substituting these values of y into equation **4)** we can get the corresponding x-values, which yields the points

$$(2, 1, 0) \text{ and } (0, -1, 0).$$

In the second case $\mu = -1$ and we can substitute this into equation (4.19) that we found before. This gives the relation

$$4y = -2x.$$

We can use this in equation **4)** to see that $y = -\frac{1}{3}$. But then, by equation **5)**, we see that necessarily $z^2 = \frac{1}{9} - 1 < 0$, which is impossible. So we get no points from this case. Therefore, we have two points with the corresponding values of f given by

$$f(2, 1, 0) = 5, \qquad f(0, -1, 0) = 1.$$

We conclude that the minimal distance from the origin along the curve of intersection is 1 and it is attained at the point $(0, -1, 0)$. You can see in Figure 4.32 that the curve of intersection is two branches of a hyperbola. We see that in this problem the points that Lagrange's method produces have a very nice geometric meaning! The two points we found correspond to points on each of the branches of the curve (which lie on the two sheets of the *hyperbolic cylinder*) which are closest to the origin.

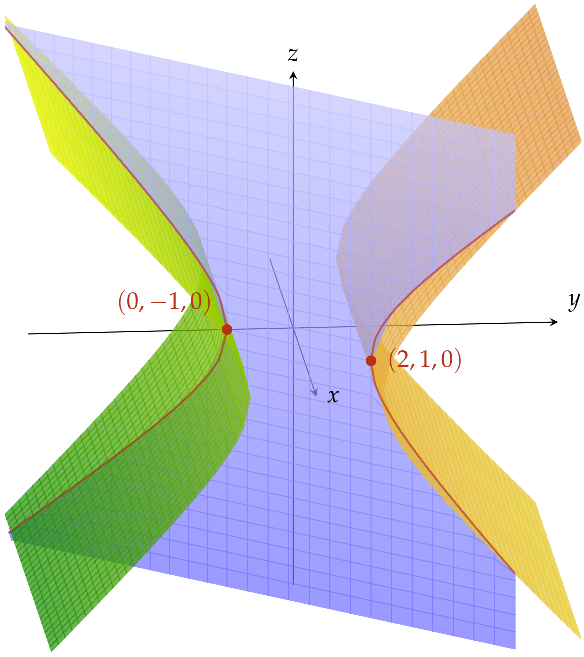


Figure 4.32: The curve of intersection of $y^2 - z^2 = 1$ and $x - y = 1$ with the points closest to the origin on both branches of the curve highlighted.

It might be tempting to eliminate one of the constraints (and one of the variables) in the previous example by substituting the constraint $z^2 = y^2 - 1$ for z^2 in $f(x, y, z)$. This would give us a function of two variables

$$f(x, y) = x^2 + 2y^2 - 1,$$

subject to one constraint $x - y - 1 = 0$. Lagrange's method would then give

$$2x - \lambda = 0, \qquad 4y + \lambda = 0, \qquad x - y - 1 = 0.$$

From the first two equations we deduce that $x = -2y$. Substituting this into the third gives $y = -1/3$, which we saw doesn't yield any solutions to the problem. So what went wrong? Where are the missing points that we found in the example above? Well, the substitution we did gives us the condition $z^2 \geq 0$, that is, $y^2 - 1 \geq 0$ on the xy-plane. We're then intersecting this set with the line $x - y = 1$, and we see that the constraint set will have endpoints on the boundary of this region, i.e. when $y^2 = 1$, see Figure 4.33. But we know from the statement of Lagrange's method that it does not work at end points (or corners) of the constraint set. Thus we have to check the case $y^2 - 1 = 0$ (that is $z^2 = 0$) separately. You can see that the coordinates of the points of intersection of the line with the region $y^2 \geq 1$ are exactly the x and y coordinates of the points we found before.

Here's one further example we didn't cover in class.

Example. Find the maximum value of

$$f(x, y, z) = xy^2z^3,$$

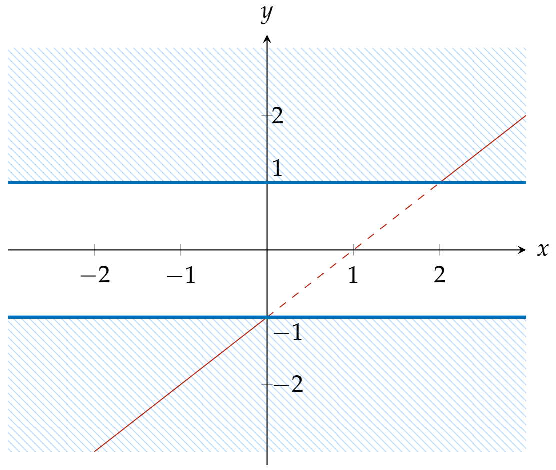


Figure 4.33: One has to be careful when trying to eliminate constraints/variables. Here we have to consider the boundary of the highlighted region separately.

subject to the conditions $x+y+z=6$, $x>0$, $y>0$, $z>0$. Notice that the last three conditions restrict us to the first octant of the three-space. In this octant we consider the constraint given by the plane $x+y+z=6$. Since all the inequalities are strict, the constraint set will have a well-defined gradient everywhere. If one of the inequalities were not strict, e.g. $x \geq 0$, then in theory we would have to consider the case of $x=0$ separately, because of the same argument as above. Of course in that case we see immediately that $f(0,y,z)=0$. In any case, we form the Lagrangian

$$L(x,y,z,\lambda) = xy^2z^3 - \lambda(x+y+z-6),$$

which gives the equations

1) $y^2z^3 - \lambda = 0$,

2) $2xyz^3 - \lambda = 0$,

3) $3xy^2z^2 - \lambda = 0$,

4) $x+y+z-6=0$.

Since $x,y,z>0$ it follows that $\lambda > 0$. We can then consider ratios of the equations:

$$\textbf{1)}\div\textbf{2)} \implies \frac{y}{2x} = 1 \iff y = 2x,$$

$$\textbf{3)}\div\textbf{2)} \implies \frac{3y}{2z} = 1 \iff z = \frac{3y}{2} = 3x.$$

Thus from the constraint we get that

$$x+y+z = x+2x+3x = 6 \implies x = 1.$$

It follows that $y=2$ and $z=3$ and the extreme value of f at this point is

$$f(1,2,3) = 108.$$

Is this actually a local max though? It could be that the point Lagrange's method provided us is a min. I'll show you two ways to determine this.

The following derivations are non-examinable. First, we compute the Hessian of $F := f - \lambda g$ (w.r.t. (x, y, z)) at the point $(1, 2, 3)$. I'll leave it as an exercise for you to show that it is equal to

$$\begin{pmatrix} 0 & 108 & 108 \\ 108 & 54 & 108 \\ 108 & 108 & 72 \end{pmatrix}.$$

Recall that the Hessian carries information about how the second partial derivatives of the function behave. By using Taylor's theorem in 3 variables for f near the point $(1, 2, 3)$ we see that

$$F(1+h, 2+k, 3+l) = F(1,2,3) + \left(\left(\begin{smallmatrix} h \\ k \\ l \end{smallmatrix}\right) \cdot \nabla\right) F(1,2,3) + \frac{1}{2!}\left(\left(\begin{smallmatrix} h \\ k \\ l \end{smallmatrix}\right) \cdot \nabla\right)^2 F(1,2,3) + \dots \tag{4.20}$$

The second order term (save for a factor of a half) is given by

$$\left(\begin{smallmatrix} h & k & l \end{smallmatrix}\right) H \left(\begin{smallmatrix} h \\ k \\ l \end{smallmatrix}\right),$$

where H is the Hessian. But since we are at a critical point of L, it means that ∇f and ∇g are parallel, so in particular the first order term in the Taylor series vanishes (because $\nabla F = \nabla f - \lambda \nabla g$). Thus if the second order term is positive near $(1, 2, 3)$ (so for small h, k, l) then we have a local *min* (since F is increasing near $(1, 2, 3)$). Similarly if the second order term is negative we get a local *max*. Why do these local extrema for F imply local extrema for f? Well, since we are at a point in the constraint set it follows that $g = 0$, and thus $F(a, b, c) = f(a, b, c)$. What about the left-hand side in (4.20)? As long as we ensure that we move within the constraint set then $F(1+h, 2+k, 3+l) = f(1+h, 2+k, 3+l)$. In the context of this problem this means that $\left(\begin{smallmatrix} h \\ k \\ l \end{smallmatrix}\right)$ is a vector that lies on the plane $x + y + z = 6$. It is then enough to find a spanning set of vectors for the plane so that we can write any vector on the plane as their linear combination. You should check that for example $\boldsymbol{u} = \left(\begin{smallmatrix} 1 \\ -1 \\ 0 \end{smallmatrix}\right)$ and $\boldsymbol{v} = \left(\begin{smallmatrix} 0 \\ 1 \\ -1 \end{smallmatrix}\right)$ span the plane (how did I come up with these? how do I know that exactly 2 vectors are enough?). This is a good exercise for your linear algebra understanding. So any vector on the constraint set is of the form $a\boldsymbol{u} + b\boldsymbol{v}$, for some (small) scalars a and b. Let's compute the Hessian in for this direction. We get

$$\begin{aligned} \begin{pmatrix} a & b-a & -b \end{pmatrix} \begin{pmatrix} 0 & 108 & 108 \\ 108 & 54 & 108 \\ 108 & 108 & 72 \end{pmatrix} \begin{pmatrix} a \\ b-a \\ -b \end{pmatrix} &= -18(9a^2 - 6ab + 5b^2) \\ &= -18((3a-b)^2 + 4b^2) < 0. \end{aligned}$$

Notice that this is strictly negative because I could express the quantity in the brackets as a sum of squares (the only way I could have 0 is if both $a = b = 0$, which is not allowed as this would correspond to the zero vector). So we see that in any possible direction (on the constraint set) at $(1, 2, 3)$, the second order terms are negative and thus f has a *local maximum*.

Here's another (often simpler, if one can do the substitution) way to figure this out. We substitute the constraint into f to get a function of two variables

$$f(x, y) = xy^2(6 - x - y)^3.$$

I'll leave it as a (slightly lengthy) exercise for you to compute that the Hessian matrix of this function at $(1, 2)$ is

$$H = \begin{pmatrix} -144 & -36 \\ -36 & -90 \end{pmatrix}.$$

Since $\det H > 0$ and $\operatorname{tr} H < 0$ it follows that we have a local maximum, as expected.

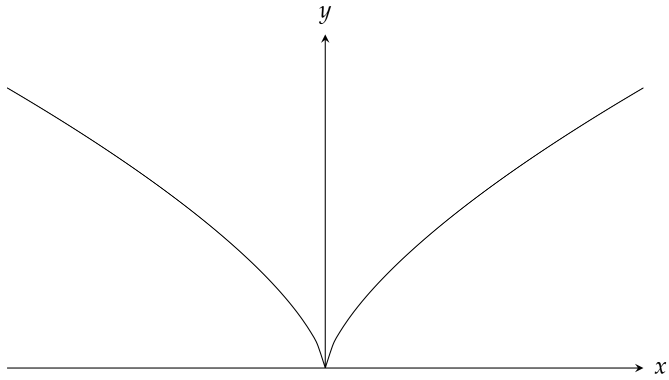


Figure 4.34: The minimal value of y along the constraint $y^3 = x^2$ occurs at the origin where the curve has a zero gradient.

Before we begin the last chapter of the course, we'll do one more example on Lagrange multipliers.

Example. Find the minimum value of

$$f(x, y) = y,$$

subject to

$$g(x, y) = y^3 - x^2 = 0.$$

Let's construct the Lagrangian:

$$L(x, y, \lambda) = y - \lambda(y^3 - x^2).$$

The critical points of L satisfy the equations

$$-2\lambda x = 0, \qquad 1 - 3\lambda y^2 = 0, \qquad y^3 - x^2 = 0.$$

From the first equation we see that $\lambda = 0$ or $x = 0$. If $\lambda = 0$ then the second eqution becomes $1 = 0$, which is a contradiction. Thus $\lambda \neq 0$ and $x = 0$. But then by the third equation $x^2 = 0$, which again gives a contradiction in the second equation. So there are no solutions! What's happening?

Certainly f is continuous everywhere, and while $g = 0$ defines an unbounded set, we still see that $y \geq 0$, so there should be a minimal value for y (i.e. for f). Let's compute the gradient of g:

$$\nabla g = \begin{pmatrix} -2x \\ 3y^2 \end{pmatrix}.$$

We notice that the gradient is $\mathbf{0}$ if $(x, y) = (0, 0)$, but this is part of the constraint set! It follows that we have a corner (actually what we call a *cusp*) at the origin, as you can see in Figure 4.34, so there is no tangent line. Clearly the origin is also the point with the smallest y-value. This is why Lagrange's method does not yield anything. This example is just to highlight the fact that you have to remember to check the points where the gradient of g is zero separately (as well as any other corners/endpoints and the points where either of the gradients of f or g don't exist).

Here's one further example I decided to add at the end of the semester. I tried to be more thorough in the explanations here.

Example. Find the points with the shortest distance to the origin from the surface

$$xyz^2 = 2.$$

As before, we want to minimise the *square* of the distance from the origin, that is, $f(x,y,z) = x^2 + y^2 + z^2$. Our constraint function is then $g(x,y,z) = xyz^2 - 2$. If you recall the statement of Lagrange's theorem, in order to be thorough, we should check whether $\nabla g = \mathbf{0}$ on the constraint curve. Crazy things can happen there, as we saw in the previous example. We have

$$\nabla g = \begin{pmatrix} yz^2 \\ xz^2 \\ 2xyz \end{pmatrix}.$$

Therefore if we set $\nabla g = \mathbf{0}$, then the third component implies that $xyz = 0$. Clearly this cannot happen on the constraint $xyz^2 = 2$, so therefore we are safe and we can proceed with the rest of the method!

Our Lagrangian function is

$$L(x,y,z,\lambda) = x^2 + y^2 + z^2 - \lambda(xyz^2 - 2).$$

Setting $\nabla L = \mathbf{0}$, we obtain the equations

1) $2x - \lambda yz^2 = 0$,

2) $2y - \lambda xz^2 = 0$,

3) $2z - 2\lambda xyz = 0$,

4) $xyz^2 - 2 = 0$.

I want to avoid doing division, so let's multiply **1)** by x and **2)** by y. We get

$$2x^2 = \lambda xyz^2, \qquad 2y^2 = \lambda xyz^2.$$

This implies that $x^2 = y^2$, from which we deduce that $x = \pm y$. Equation **3)** factorises as $2z(1 - \lambda xy) = 0$. This means that either $z = 0$ or $\lambda xy = 1$. But if $z = 0$ then this contradicts **4)**. Similarly we can see that $x \neq 0$ and $y \neq 0$. Therefore we find that

$$\lambda = \frac{1}{xy}.$$

Substituting this back into **1)** shows that

$$2x - \frac{z^2}{x} = 0,$$

which rearranges to $2x^2 = z^2$. We can substitute this to **4)** to get

$$2x^3y = 2. \tag{4.21}$$

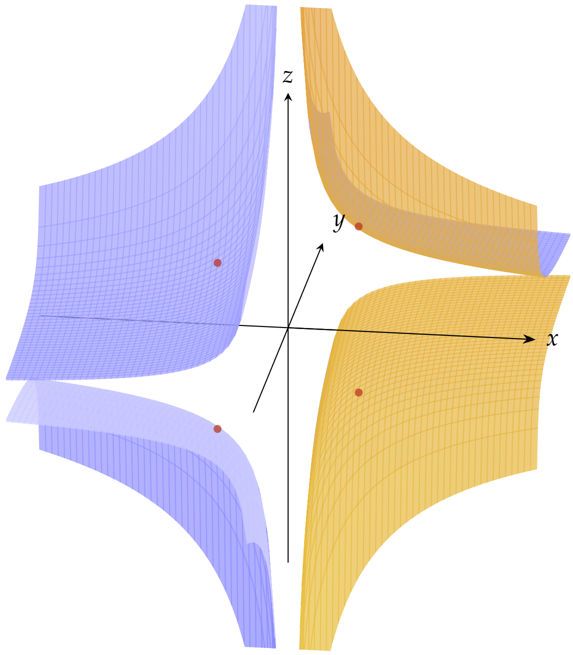


Figure 4.35: The four closest points to the origin on the surface $xyz^2 = 2$.

We know that $x = \pm y$, but if we try to put $x = -y$ in (4.21), then we see that we get a contradiction (why?). Therefore $x = y$ and $x^4 = 1$, which means that $x = y = \pm 1$. For these two values of x and y we can find the value of z from $2x^2 = z^2$, that is, $z = \pm\sqrt{2}x$. Therefore we get four points:

$$(1, 1, \pm\sqrt{2}), (-1, -1, \pm\sqrt{2}).$$

Each of these is at a distance $\sqrt{1 + 1 + 2} = 2$ from the origin. You can see these four points on the four disjoin sheets of the surface in Figure 4.35.

Chapter 5

Multiple Integrals

5.1 Double Integrals

Recommended problems from §15.1: **29, 31, 47, 49**.

In this last chapter of the course we will learn how to integrate multivariable functions. We start with the easiest case of integrating a function $f(x, y)$ of two variables over some bounded domain D on the xy-plane ($\mathbb{R}^2$). Before we jump in, let's quickly remind ourselves how integrals of single variable functions were defined. Consider the integral

$$\int_a^b f(x)dx.$$

First we divide the interval $[a, b]$ into n subintervals. We do this by picking $n + 1$ points in the interval (always including the endpoints), that is, we pick x_i such that

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

Then in each of these subintervals we pick a sample point $\tilde{x}_i$, where $\tilde{x}_i \in [x_i, x_{i+1}]$ for $i = 0, \ldots, n - 1$. We do this again as n keeps getting bigger and bigger, but we have to ensure that as we increase n then the length of the largest subinterval for each n gets smaller and smaller and tends to 0 as $n \to \infty$. That is, we require that

$$\lim_{n\to\infty} \max_{i=0,\ldots,n-1} (x_{i+1} - x_i) = 0.$$

We then construct rectangles on the xy-plane of width $x_{i+1} - x_i$ and height $f(\tilde{x}_i)$, which is depicted in Figure 5.1. From the picture we can see that as the length of these intervals keeps getting smaller and smaller (i.e. the width of the rectangles), then the total area of the rectangles gets closer and closer to the area under the curve, which we know is the integral of f. Thus we define the integral (if the limit exists) as

$$\int_a^b f(x) = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(\tilde{x}_i)(x_{i+1} - x_i). \tag{5.1}$$

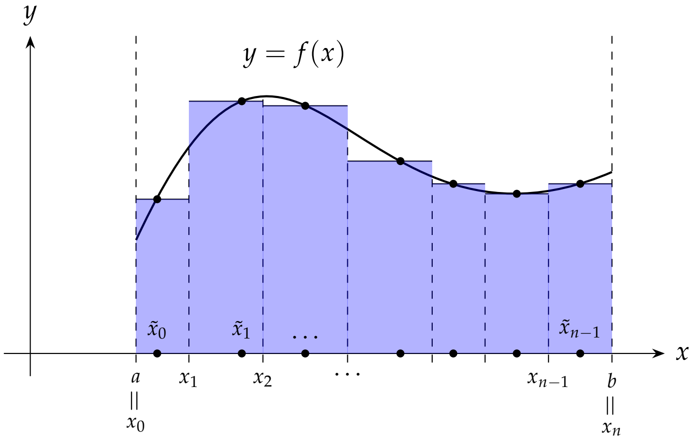


Figure 5.1: Definition of the integral $\int_a^b f(x)dx$.

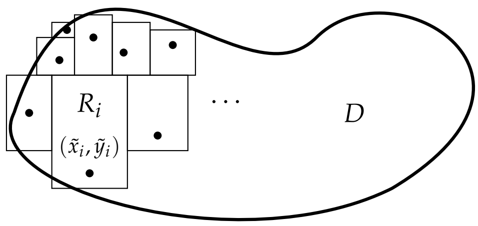


Figure 5.2: Covering of a domain in $\mathbb{R}^2$ by rectangles R_i and sample points $(\tilde{x}_i, \tilde{y}_i) \in R_i$.

Notice that here on the right-hand side we are just summing up the area of each of the individual rectangles. Sums like this are called **Riemann sums**[1].

Now, let's take a function $f(x, y)$ of two variables on a bounded domain $D \subsetneq \mathbb{R}^2$ (there are some assumptions on the set D, especially its boundary, but we won't worry about the details in this course). This set is generally much more complicated than the interval of integration in the one variable case. The idea is that we cover the domain D by n rectangles, which we denote by R_i, as in Figure 5.2. As before, we require that the rectangles get smaller as $n \to \infty$, or more precisely that

$$\lim_{n\to\infty} \max_{i=0,\ldots,n-1} \operatorname{area}(R_i) = 0.$$

From each rectangle we pick a sample point, say, $(\tilde{x}_i, \tilde{y}_i) \in R_i$. We can then define integration over this set in much the same way as we did for single variable functions.

[1] Another equivalent, but perhaps more convenient, definition for the Riemann integral is through the supremum and infimum of f in each subinterval. The corresponding sums are called **Darboux sums**.

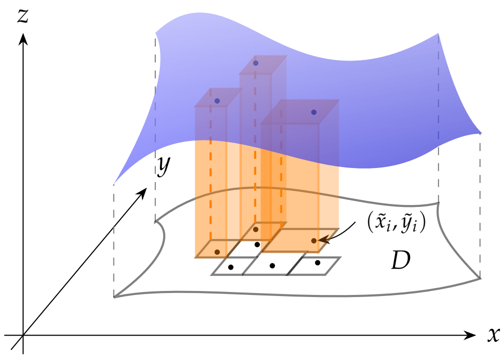


Figure 5.3: Double integrals are defined by considering the volume between the graph of f and the xy-plane.

We define the **double integral** of $f(x, y)$ over the region D as the limit (if it exists)

$$\iint_D f(x,y)dA = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(\tilde{x}_i, \tilde{y}_i)\,\text{area}(R_i).$$

The sum on the right-hand side carries an obvious geometric meaning. Each summand gives the volume of a "box" with base R_i and height $f(\tilde{x}_i, \tilde{y}_i)$, as we see in Figure 5.3. Therefore, just as for a single variable function its integral gives the *signed* area under its graph, the double integral of $f(x, y)$ gives the *signed*[2] volume between the graph of $f(x, y)$ (which is the surface $z = f(x, y)$) and the xy-plane. The double integral also satisfies all the usual properties:

i) We have linearity: for $\alpha, \beta \in \mathbb{R}$

$$\iint_D (\alpha f(x,y) + \beta g(x,y))dA = \alpha \iint_D f(x,y)dA + \beta \iint_D g(x,y)dA.$$

ii) If D is a union of two disjoint sets D_1 and D_2 (that is $D_1 \cap D_2 = \emptyset$) then

$$\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA.$$

So how do we evaluate double integrals? Just as in one variable, the definition of the integral is very cumbersome for calculations. Instead, we'll use the following observation. Suppose we are integrating $f(x, y)$ over a rectangle $R = [a, b] \times [c, d]$. Let's for the sake of exposition suppose that f is positive. We partition the interval $[a, b]$ into n subintervals by selecting points $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$. This partitions R into strips of the form $[x_i, x_{i+1}] \times [c, d]$, see Figure 5.4. Now in each of these strips we consider the area under the surface at the left endpoint of the strip. The area of this kind of sheet is just the integral $\int_c^d f(x_i, y)dy$. For each of these strips we then associate a solid with base given by the strip and the top given by extending the curve $z = f(x_i, y)$ from x_i to x_{i+1}. The volume of this solid is naturally $(x_{i+1} - x_i)\int_c^d f(x_i, y)dy$. Therefore,

[2]Signed means that the portion of the volume that is below the xy-plane is taken with a negative sign.

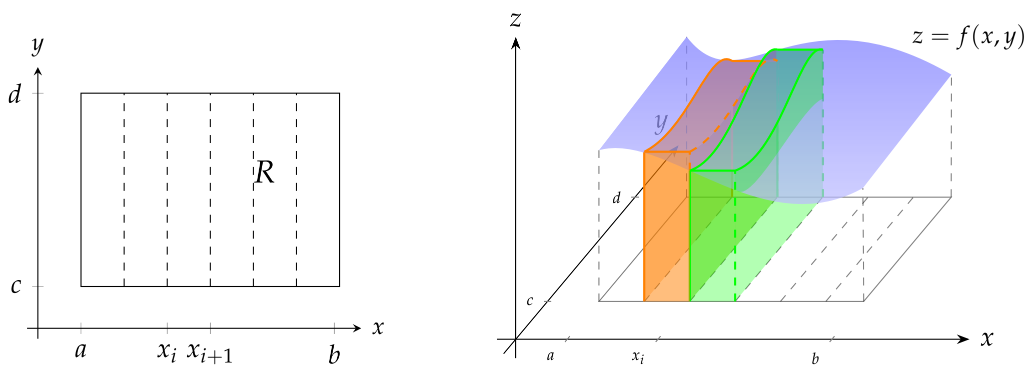


Figure 5.4: The derivation of Fubini's theorem. On the left we see the domain of integration divided into strip and on the right the three dimensional solids corresponding to each of the strips. The total volume of the solids approximates the value of the integral $\iint_R f(x,y)dA$.

we can approximate the volume under the surface $z = f(x,y)$ (that is, the integral of f over R) by adding up the volume of each of these solids, which gives

$$\iint_D f(x,y)dA \approx \sum_{i=0}^{n-1}(x_{i+1}-x_i)\int_c^d f(x_i,y)dy.$$

But now this resembles the definition of the single integral we saw at the beginning (5.1) (if we treat the whole integral w.r.t. y as a function of x)! Therefore, if we take the limit as $n \to \infty$ we arrive at

$$\iint_D f(x,y)dA = \int_a^b \left(\int_c^d f(x,y)dy\right) dx.$$

Here we have to assume that all the integrals exist so that the limits converge. For us this means that we have to assume that f is continuous on R. We can of course do this whole business by looking at strips on R in the other direction (by partitioning $[c,d]$), which reverses the order of integrals. This derivation leads to the following theorem.

Fubini's Theorem (for bounded rectangles). *Let $R = [a,b] \times [c,d]$ and suppose that $f(x,y)$ is a continuous function on R. Then*

$$\begin{aligned}\iint_R f(x,y)dA &= \int_a^b \int_c^d f(x,y)dy\,dx \\ &= \int_c^d \int_a^b f(x,y)dx\,dy.\end{aligned}$$

*The integrals on the right are called **iterated integrals**.*

So Fubini's theorem is telling us that to compute the double integral of f over some rectangle, we can just do the integration with respect to each of the variables one at a time. The above immediately implies that if $f(x,y)$ can be written as a product of functions depending only on x or y, i.e. $f(x,y) = g(x)h(y)$ then

$$\int_a^b \int_c^d g(x)h(y)dy\,dx = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right).$$

Let's do an example.

Example. Suppose that $R = [0,1] \times [0,1]$ and $f(x,y) = (x+y)^2$. We observe that f is continuous on R so Fubini's theorem applies. I'll show you two ways to do this. The first one is just by pure evaluation, the second way is a bit *ad hoc*, but it can be made more obvious (by doing a substitution) after we learn some more material.

By Fubini's theorem,

$$\iint_R f(x,y)dA = \int_{x=0}^1 \int_{y=0}^1 (x+y)^2 dy\,dx. \tag{5.2}$$

Here I've written the variables of integration in the lower limit to emphasise the order of integration. Strictly speaking this is superfluous as you should always have the differentials at the end (dy and dx) to be in the correct order anyway (i.e. the innermost integral corresponds to the innermost differential). Therefore, I won't be writing these "helpers" down too often and instead recommend you to be precise with your notation. Typically, the simpler notation one needs to use, the more readable the math becomes. Anyway, back to evaluating (5.2). We start by expanding the square:

$$\begin{aligned}\int_{x=0}^1 \int_{y=0}^1 (x+y)^2 dy\,dx &= \int_0^1 \int_0^1 \left(x^2+2xy+y^2\right) dy\,dx \\ &= \int_0^1 \left(\int_0^1 x^2 dy + \int_0^1 2xy\,dy + \int_0^1 y^2 dy\right) dx.\end{aligned}$$

The inner integrals are over y so we can treat x as a constant when we integrate and get

$$\begin{aligned}&= \int_0^1 \left(\left[x^2y\right]_{y=0}^1 + \left[xy^2\right]_{y=0}^1 + \left[\tfrac{1}{3}y^3\right]_{y=0}^1\right) dx \\ &= \int_0^1 \left(x^2+x+\tfrac{1}{3}\right) dx \\ &= \left[\frac{x^3}{3}+\frac{x^2}{2}+\frac{x}{3}\right]_0^1 \\ &= \frac{1}{3}+\frac{1}{2}+\frac{1}{3} = \frac{7}{6}.\end{aligned}$$

On the other hand, we can do the integration directly by realising that

$$\frac{\partial}{\partial y}\frac{1}{3}(x+y)^3 = (x+y)^2.$$

Therefore

$$\begin{aligned}\int_0^1 \int_0^1 (x+y)^2 dy\,dx &= \int_0^1 \left[\frac{1}{3}(x+y)^3\right]_{y=0}^1 dx \\ &= \frac{1}{3}\int_0^1 \left((x+1)^3 - x^3\right) dx,\end{aligned}$$

and then proceed with the evaluation as before. Practically speaking, you could arrive at this method by doing the substitution $u = y + x$ in the first integral, so that u goes from x to $x + 1$, but this corresponds to integrating over a region which is not a rectangle, and we haven't learnt about those yet.

We learnt last time that when evaluating double integrals, we can exchange the order of integration for the iterated integrals by Fubini's theorem. Sometimes one order is easier than the other. Later, once we do integrals over general regions, we will see examples that are impossible to do without exchanging the order! For now, let's consider a simpler example.

Example. Let $R = [0,1] \times [0,1]$ and

$$f(x,y) = \frac{x}{1+xy}.$$

We want to evaluate the integral of f over R. Clearly f is continuous on R so Fubini's theorem applies. But in which order should we iterate the integral? Well, in this example either way is doable, but since the fraction is simpler in terms of y (it only appears in the denominator, as opposed to x), it makes sense to try and integrate first with respect to y and then with respect to x. We obtain

$$\begin{aligned}\iint_R f(x,y)dA &= \int_{x=0}^1 \int_{y=0}^1 \frac{x}{1+xy} dy\, dx \\ &= \int_0^1 x \left(\int_0^1 \frac{1}{1+xy} dy \right) dx,\end{aligned}$$

since x is just a constant inside the integral with respect to y. To compute the inner integral we notice that

$$\frac{\partial}{\partial y}\left(\frac{1}{x}\ln(1+xy)\right) = \frac{1}{1+xy}.$$

Again, you could make this more obvious by doing a substitution in the integral over y, but we refrain from this for now for the same reason as last time. Thus

$$\begin{aligned}\iint_R f(x,y)dA &= \int_0^1 x \left[\frac{1}{x}\ln(1+xy)\right]_{y=0}^1 dx \\ &= \int_0^1 \ln(1+x)dx.\end{aligned}$$

To integrate this, first we do a substitution $1 + x = u$ so that $dx = du$, which gives

$$\iint_R f(x,y)dA = \int_1^2 \ln u\, du.$$

Then, we all know how to integrate ln (by parts), to get

$$\begin{aligned}\iint_R f(x,y)dA &= [u \ln u]_1^2 - \int_1^2 du \\ &= 2\ln 2 - 1.\end{aligned}$$

Next, we want to investigate what happens if we allow the domain of integration for double integrals to be undounded (e.g. $R = [0,\infty) \times [0,\infty)$). We need to be a bit careful here. Consider the double sequence a_{ij}, which is defined by

$$a_{ij} = \begin{cases} 1, & \text{if } i = j, \\ -1, & \text{if } i+1 = j, \\ 0, & \text{otherwise.} \end{cases}$$

We can visualise this sequence as an "infinite grid"

$$\begin{array}{cccccc} 1 & -1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & -1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

where the entry on the i-th row in the j-th column corresponds to a_{ij}. Now we want to sum all of these numbers together. Two obvious ways of doing this is to either first sum up all the entries in each of the rows separately (and obtain what we call *row sums*) and then add the row sums together, or the other way around, where we first add entries in the columns together (*column sums*) and then sum up those. These can be written as

$$R = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}, \qquad C = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

Let's evaluate these sums. First, for the row sums we compute that each of them is equal to 0, and thus $R = 0$. On the other hand, for the column sums we see that the first one is 1 while the remaining ones are all 0, and thus $C = 1$. In terms of the above diagram this becomes

$$\begin{array}{cccccc||c} 1 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \hline\hline 1 & 0 & 0 & 0 & 0 & \cdots & 1 \diagdown 0 \end{array}$$

In particular $C \neq R$, so if we exchange the order of summation the value of the sum changes! This is because the sums don't converge *absolutely*, that is, the sum

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}|$$

diverges. A similar phenomenon happens with integrals. To avoid this, we need to require absolute convergence of our integrals.

Fubini's Theorem (for unbounded rectangles). *Let $R = [a, b] \times [c, d]$, where a, b, c, d are possibly infinite. Suppose that one of the integrals*

$$\int_a^b \int_c^d |f(x,y)|\,dy\,dx, \qquad \int_c^d \int_a^b |f(x,y)|\,dx\,dy$$

exists (i.e. is finite). Then Fubini's theorem works as before:

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

Example. Let $R = [0, \infty) \times [0, 2\pi]$ and $f(x,y) = e^{-x}\sin y$. To apply Fubini we have to check whether f is absolutely integrable. We look at

$$\int_0^{\infty} \int_0^{2\pi} |f(x,y)|\,dy\,dx.$$

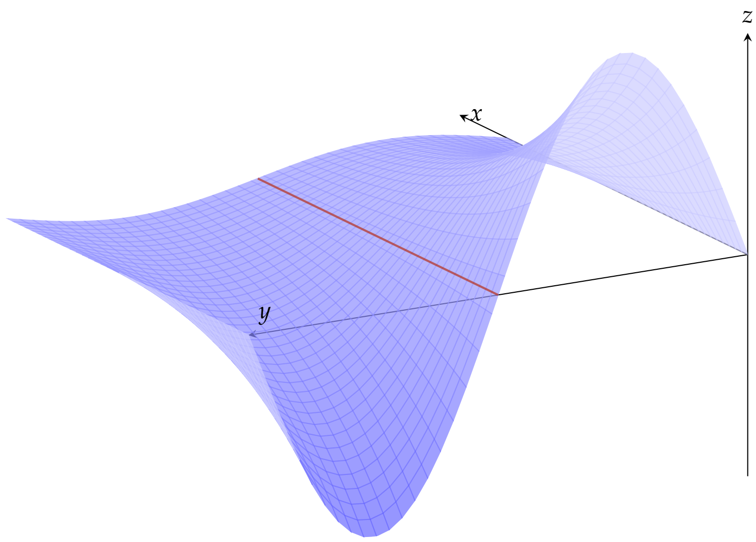


Figure 5.5: The surface $z = e^{-x}\sin y$ has two equal parts in the domain $[0,\infty)\times[0,2\pi]$, one lying above the xy-plane and one below. Therefore the integral is 0.

To evaluate the inner integral we need to evaluate

$$\int_0^{2\pi} |\sin y| dy.$$

We know that in the interval $[0,\pi]$ the sine is positive, whereas in the interval $(\pi, 2\pi]$ it is negative. Thus

$$\int_0^{2\pi} |\sin y| dy = \int_0^{\pi} \sin y\, dy + \int_{\pi}^{2\pi} (-\sin y) dy = [-\cos y]_0^{\pi} - [-\cos y]_{\pi}^{2\pi} = 4.$$

Therefore

$$\int_0^{\infty}\int_0^{2\pi} |e^{-x}\sin y| dy\, dx = 4\int_0^{\infty} e^{-x} = 4[-e^{-x}]_0^{\infty} = 4(e^0 - \lim_{x\to\infty} e^{-x}) = 4.$$

Therefore our integral converges absolutely and Fubini's theorem gives (notice that f is written as a product of functions which only depend on one of the variables each)

$$\iint_R f(x,y) dA = \int_0^{\infty} e^{-x} dx \int_0^{2\pi} \sin y\, dy = 0,$$

since we are integrating $\sin y$ over a full period. It's crucial to notice that the answer is different than what we got for the integral of $|f(x,y)|$. This is because the sine function gives two equal parts to the surface (symmetric about the $y = \pi$ plane) with opposite sign, as you can see in Figure 5.5.

Now let's do an example where Fubini's theorem doesn't work. There's a quick and a long way to do the following computations. I showed the quick way during the lecture due to lack of time, but it's not something you could come up by yourself (so don't worry)! On the other hand, the long way is, well, pretty long. Therefore I decided to put in in a separate note that you can find on myCourses. You can take a look at it if you want, although it's not mandatory (on the other hand, while it's a long derivation, there's nothing I've used that you don't know, so technically you should be able to do it, too). Notice that I've changed the notation a little bit since some people found what I used during the lectures confusing.

Example. Let

$$f(x,y) = \frac{y^2 - x^2}{(x^2+y^2)^2},$$

and let $R = [0,1] \times [0,1]$. We want to compare the two different iterated integrals of f corresponding to this region of integration, namely,

$$\int_0^1 \int_0^1 f(x,y)dy\,dx, \qquad \int_0^1 \int_0^1 f(x,y)dx\,dy.$$

To do this it is helpful to notice that

$$\frac{\partial}{\partial x} \arctan \frac{y}{x} = \frac{-y/x^2}{1 + \left(\frac{y}{x}\right)^2} = \frac{-y}{x^2+y^2}.$$

Now differentiate with respect to y:

$$\frac{\partial^2}{\partial y \partial x} \arctan \frac{y}{x} = \frac{\partial}{\partial y} \frac{-y}{x^2+y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = f(x,y).$$

We can use this information to evaluate our integrals quickly through the Fundamental Theorem of Calculus. Recall that it says that if $G(t)$ is an antiderivative of $g(t)$ (so $G'(t) = g(t)$) then $\int_a^b g(t)dt = G(b) - G(a)$.

$$\int_0^1 \int_0^1 f(x,y)dy\,dx = \int_0^1 \int_0^1 \frac{\partial^2}{\partial y \partial x} \arctan \frac{y}{x} dy\,dx \tag{5.3}$$

$$= \int_0^1 \left[\frac{\partial}{\partial x} \arctan \frac{y}{x}\right]_{y=0}^1 dx \tag{5.4}$$

$$= \int_0^1 \left(\frac{\partial}{\partial x} \arctan \frac{1}{x} - 0\right) dx. \tag{5.5}$$

Next[3], we just evaluate the remaining integral over x by Fundamental Theorem of Calculus. Therefore

$$\begin{aligned}\int_0^1 \int_0^1 f(x,y)dy\,dx &= \left[\arctan \frac{1}{x}\right]_0^1 \\ &= \arctan 1 - \lim_{x\to 0^+} \arctan \frac{1}{x} \\ &= \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}.\end{aligned}$$

[3] I'm skipping over one technicality here. Strictly speaking in our region of integration x is allowed to be zero which causes problems in the denominator. However, turns out that this is not an issue if you treat the integrals as *improper integrals*.

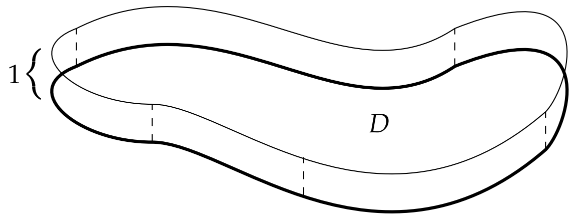


Figure 5.6: To compute the area of D we can compute the volume of the cylinder with base D and height 1.

Now, let's look at the other iterated integral. With a little trick we can use the derivation above to find its value easily:

$$\begin{aligned}\int_0^1 \int_0^1 f(x,y)dx\,dy &= \int_{y=0}^1 \int_{x=0}^1 \frac{y^2-x^2}{(x^2+y^2)^2}dx\,dy \\ &= -\int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dx\,dy.\end{aligned}$$

But now this iterated integral is exactly in the form that we already evaluated (save for the minus sign in front). We've just interchanged the symbols x and y. Therefore it follows immediately that

$$\int_0^1 \int_0^1 f(x,y)dx\,dy = -\left(-\frac{\pi}{4}\right) = \frac{\pi}{4}.$$

So we see that the two iterated integrals yield different answers. Why does this not contradict Fubini's theorem? This is because our function is not absolutely integrable over the region R, that is,

$$\int_0^1 \int_0^1 |f(x,y)|dy\,dx = \infty.$$

Finally, we define the **mean value** (or **average value**) of a function $f(x,y)$ over a domain D. This is

$$\overline{f} = \frac{\iint_D f(x,y)dA}{\text{area } D} = \frac{\iint_D f(x,y)dA}{\iint_D 1\,dA}.$$

To see the last equality, we note that if we integrate 1 over some bounded domain $D \subsetneq \mathbb{R}^2$, then this gives exactly the area of D. This is because by the definition of the double integral, we are computing the volume under the surface $z=1$ (which is a plane parallel to the xy-plane) restricted to the domain D. This solid is of course a cylinder with base D and height 1, see Figure 5.6. Therefore, we have that

$$\begin{aligned}\iint_D 1\,dA &= \text{volume of cylinder with base } D \text{ and height } 1 \\ &= (\text{area of base}) \times (\text{height}) \\ &= \text{area } D.\end{aligned}$$

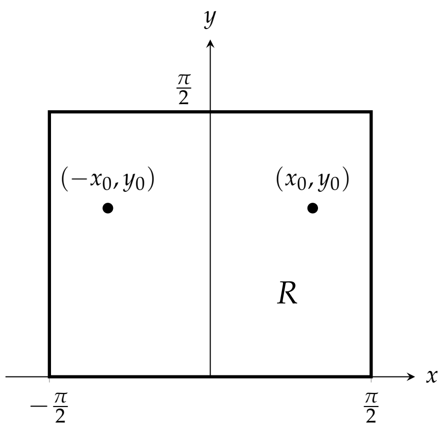


Figure 5.7: The rectangle R is symmetric about the y-axis.

Example. Let's compute the mean value of

$$f(x, y) = 2\cos x \, \sin y$$

over the region $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$. First we compute the integral in the numerator of the mean value (f is continuous):

$$\begin{aligned}\iint_R f(x,y)dA &= 2\int_0^{\pi/2} \cos x \, dx \int_0^{\pi/2} \sin y \, dy \\ &= 2[-\sin x]_0^{\pi/2}[\cos y]_0^{\pi/2} \\ &= -2(1-0)(0-1) = 2.\end{aligned}$$

The area of the rectangle R is of course $\left(\frac{\pi}{2}\right)^2$. Therefore the mean value of f is

$$\overline{f} = \frac{2}{\pi^2/4} = \frac{8}{\pi^2}.$$

Let's change the integral slightly to the following:

$$\int_{y=0}^{\pi/2} \int_{x=-\pi/2}^{\pi/2} x\cos x \, \sin y \, dx \, dy.$$

Then we can actually immediately say that the above integral is equal to 0! Why is this? The integrand is an odd function of x (since $\cos x$ is even), so the integrand is "anti-symmetric" about the line $x = 0$. On the other hand, the domain of integration $R = [-\frac{\pi}{2}, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$ is also symmetric about $x = 0$ (i.e. it can be reflected about the y-axis), as you can see in Figure 5.7. Therefore, for any point on the right side of the rectangle, (x_0, y_0), there is a unique corresponding point on the left side of the rectangle, $(-x_0, y_0)$ where f takes an equal but opposite value, that is, $f(-x_0, y_0) = -f(x_0, y_0)$. Therefore, the integral over the left part of the rectangle is equal to the integral over the right part, but with the opposite sign. You can see this in terms of the graph of f in Figure 5.8 (although this visualisation is unnecessary to see that the integral vanishes).

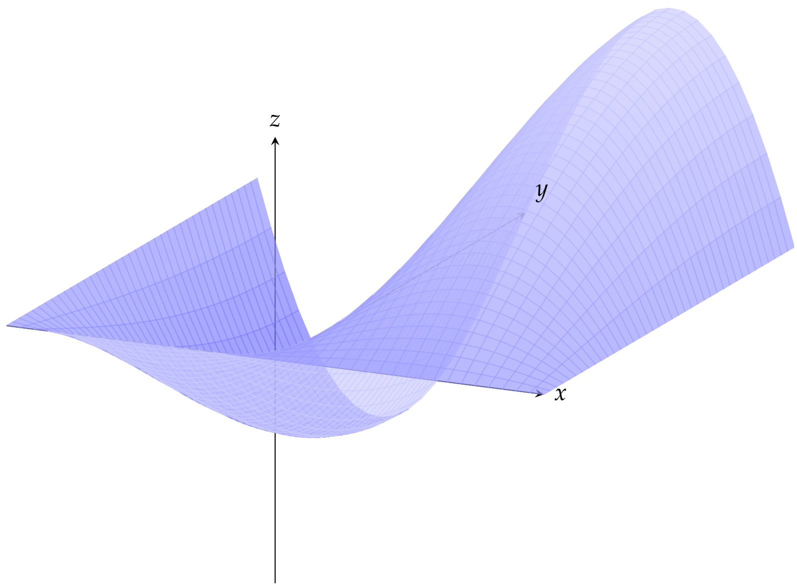


Figure 5.8: The graph of the function $f(x, y) = x \cos x \, \sin y$ restricted to $R = [-\frac{\pi}{2}, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$.

Aside: Nasty Integral

Let
$$f(x,y) = \frac{y^2 - x^2}{(x^2+y^2)^2},$$
and let $R = [0,1] \times [0,1]$. We want to evaluate the following nasty iterated integral:
$$\begin{aligned} \int_0^1 \int_0^1 \frac{y^2 - x^2}{(x^2+y^2)^2} dy\, dx &= \int_0^1 \int_0^1 \frac{y^2 + x^2 - 2x^2}{(x^2+y^2)^2} dy\, dx \\ &= \int_0^1 \int_0^1 \left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} \right) dy\, dx. \end{aligned} \tag{5.6}$$

Let's evaluate the two parts of the inner integral separately.
$$\begin{aligned} \int_0^1 \frac{dy}{x^2+y^2} &= \frac{1}{x^2} \int_0^1 \frac{dy}{1 + \left(\frac{y}{x}\right)^2} \\ &= \frac{1}{x} \left[\arctan \frac{y}{x} \right]_0^1 \\ &= \frac{1}{x} \arctan \frac{1}{x}, \end{aligned} \tag{5.7}$$
since $\arctan 0 = 0$ and
$$\frac{\partial}{\partial y} \arctan \frac{y}{a} = \frac{1}{a} \frac{1}{1 + (\frac{y}{a})^2}.$$
On the other hand, to evaluate
$$\int_0^1 \frac{dy}{(x^2+y^2)^2} = \frac{1}{x^4} \int_0^1 \frac{dy}{\left(1 + \frac{y^2}{x^2}\right)^2}, \tag{5.8}$$
we consider the integral
$$I = \int \frac{dy}{\left(1 + \frac{y^2}{a^2}\right)^2}.$$
First, do a substitution $y = a \tan u$, so that $dy = a \sec^2 u\, du$. Thus
$$I = \int \frac{a \sec^2 u}{(1 + \tan^2 u)^2} du.$$
But since $\sin^2 u + \cos^2 u = 1$, it follows that $\tan^2 u + 1 = \sec^2 u$. Using this identity in I gives
$$I = a \int \frac{\sec^2 u}{\sec^4 u} du = a \int \frac{1}{\sec^2 u} du = a \int \cos^2 u\, du.$$
But we all know how to integrate cosine squared. We use the double angle formula to write
$$I = a \int \frac{1}{2} (1 + \cos 2u)\, du = \frac{a}{2} \left(u + \frac{1}{2} \sin 2u \right) + C.$$
Now we just need to substitute back for u. Clearly, $u = \arctan \frac{y}{a}$. On the other hand, if $\tan u = \frac{y}{a}$, it follows that $\sin u = \frac{y}{\sqrt{a^2+y^2}}$ and $\cos u = \frac{a}{\sqrt{a^2+y^2}}$ (if you are unsure about

this, draw the triangle!). Therefore, since $\sin 2u = 2\sin u \cos u$, we have derived the identity

$$I = \int \frac{dy}{(1+\frac{y^2}{a^2})^2} = \frac{a}{2}\left(\arctan\frac{y}{a} + \frac{ay}{a^2+y^2}\right) + C.$$

It follows that

$$\begin{aligned}\int_0^1 \frac{dy}{(1+\frac{y^2}{a^2})^2} &= \frac{a}{2}\left[\arctan\frac{y}{a} + \frac{ay}{a^2+y^2}\right]_0^1 \\ &= \frac{a}{2}\left(\arctan\frac{1}{a} + \frac{a}{a^2+1}\right).\end{aligned}$$

We use this in (5.8) with $a = x$ to get

$$\int_0^1 \frac{dy}{(x^2+y^2)^2} = \frac{1}{x^4}\frac{x}{2}\left(\arctan\frac{1}{x} + \frac{x}{x^2+1}\right).$$

Therefore substituting the above and (5.7) into (5.6) gives

$$\begin{aligned}\int_0^1\int_0^1 \frac{y^2-x^2}{(x^2+y^2)^2}dy\,dx &= \int_0^1 \left(\frac{1}{x}\arctan\frac{1}{x} - 2x^2\frac{1}{2x^3}\left(\arctan\frac{1}{x} + \frac{x}{x^2+1}\right)\right)dx \\ &= \int_0^1 \frac{-1}{x^2+1}dx \\ &= -\left[\arctan x\right]_0^1 = -\frac{\pi}{4},\end{aligned}$$

which is the same answer as we got with the other method! Now you probably understand why I didn't evaluate the integral this way during the lecture.

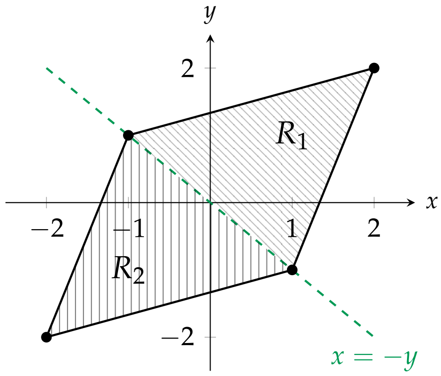


Figure 5.9: The parallelogram R divided along its axis of symmetry into two congruent parts, R_1 and R_2.

Here's one more example of using symmetry to evaluate a double integral.

Example. Consider the function

$$f(x, y) = x + y,$$

on the parallelogram R with vertices $(2, 2)$, $(-1, 1)$, $(-2, -2)$, $(1, -1)$. Then we instantly know that

$$\iint_R (x + y)dA = 0.$$

Why is this? I left it to you as an exercise during the class to figure it out, if you haven't thought about it yet please do so first (maybe the above picture helps). First of all we notice that R is symmetric about the line $x = -y$ (of course it is also symmetric about $x = y$, either of these choices works). This line of symmetry divides the parallelogram into two congruent halves, the upper half R_1 and the lower half R_2, as shown in Figure 5.9. Therefore, for each point $(x_0, y_0) \in R_1$, there is a corresponding point $(-x_0, -y_0) \in R_2$ (when reflected about the origin). Moreover, the function $f(x, y)$ is anti-symmetric about this line, that is,

$$\begin{aligned} f(-x_0, -y_0) &= -x_0 - y_0 \\ &= -(x_0 + y_0) \\ &= -f(x_0, y_0). \end{aligned}$$

Therefore the integral of f over R_1 will be equal to the integral of f over R_2, but with opposite sign, so that

$$\begin{aligned} \iint_R f(x, y)dA &= \iint_{R_1} f(x, y)dA + \iint_{R_2} f(x, y)dA \\ &= \iint_{R_1} f(x, y)dA - \iint_{R_1} f(x, y)dA \\ &= 0, \end{aligned}$$

as we wanted.

It should be noted that in general it's not often easy to find/notice these kind of symmetries and that one can always rely on the "less flashy" way of evaluating the integral with Fubini's theorem. As an exercise, use Fubini's theorem to prove that the above integral really vanishes.

5.2 Double Integrals Over General Regions

Recommended problems from §15.2: **7, 10, 25, 29, 31**.

We'll be working with domains on $\mathbb{R}^2$ of two possible types, by either specifying two curves as a function of x or as a function of y. There can be more complicated situations, but we can always bring everything back to either of these forms by splitting the domain into parts. In the first case we have x varying as a "free variable" and y bounded between two curves given as functions of x. This can be described more concretely as the domain

$$D_1 = \{(x,y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\},$$

which could look something like in Figure 5.10.

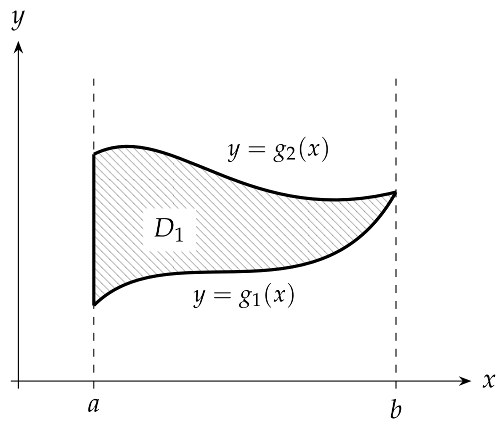


Figure 5.10: A domain D_1 between the lines $x = a$ and $x = b$ and the curves $y = g_1(x)$ and $y = g_2(x)$.

On the other hand, we can also have y as the free variable and give bound for x in terms of functions of y to get

$$D_2 = \{(x,y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\},$$

which could look like in Figure 5.11.

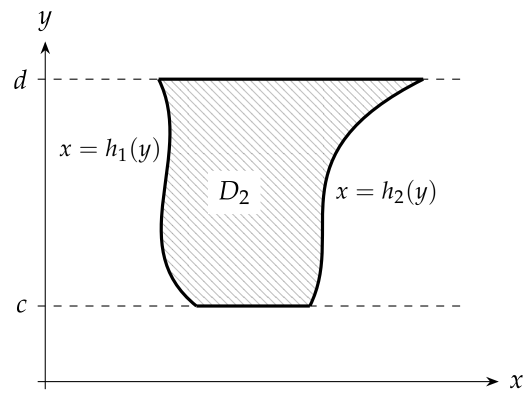


Figure 5.11: A domain D_2 between the lines $y = c$ and $y = d$ and the curves $x = h_1(y)$ and $x = h_2(y)$.

Then, if we want to integrate $f(x, y)$ over, say, D_1, we can iterate the integral as (assuming the integrals exist)

$$\iint_{D_1} f(x,y)dA = \int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} f(x,y)dy\,dx,$$

and similarly over D_2

$$\iint_{D_2} f(x,y)dA = \int_{y=c}^{d} \int_{x=h_1(y)}^{h_2(y)} f(x,y)dx\,dy.$$

Sometimes it's possible to parametrise D in either of these two ways. Both ways of parametrising should of course yield the same result (as long as Fubini's theorem applies, i.e. if the integrals converge absolutely). Often one way is easier than the other, it takes experience to realise which one it is (this is a subtle hint for you to work through many exercises!).

Example. Let

$$f(x,y) = \frac{y}{x^2+1},$$

and $D = \{(x, y) : 0 \leq x \leq 4, 0 \leq y \leq \sqrt{x}\}$. It's often helpful to visualise the region of integration (even if it is not necessary for every question). You can see the domain for this question in Figure 5.12. The integrand is continuous everywhere so Fubini's theorem applies (since the region of integration is bounded). We can then iterate as

$$\begin{aligned}\iint_D f(x,y)dA &= \int_{x=0}^{4} \int_{y=0}^{\sqrt{x}} \frac{y}{x^2+1} dy\,dx \\ &= \int_0^4 \frac{\left[\frac{y^2}{2}\right]_0^{\sqrt{x}}}{x^2+1} dx \\ &= \frac{1}{2}\int_0^4 \frac{x}{x^2+1} dx. \end{aligned} \tag{5.9}$$

As a quick remark, we notice that the integrand is of the form $\frac{g'(x)}{g(x)}$ and we know that $\int \frac{g'(x)}{g(x)} dx = \ln g(x) + C$, so we could find the value of the above integral directly.

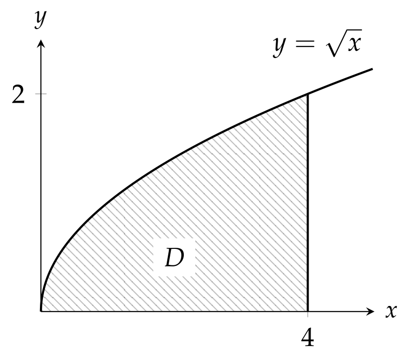


Figure 5.12: The region of integration $D = \{(x, y) : 0 \le x \le 4, 0 \le y \le \sqrt{x}\}$.

However, it's perfectly fine to take your time and do a substitution $x^2 + 1 = u$ so that $2x\, dx = du$. Then

$$\begin{aligned}\iint_D f(x,y)dA &= \frac{1}{4}\int_1^{17} \frac{1}{u} du \\ &= \frac{1}{4}[\ln u]_1^{17} \\ &= \frac{1}{4}\ln 17.\end{aligned}$$

We can also do the integration the other way around (though this will be a tad more tedious). For this we need to parametrise the domain as functions of y. From Figure 5.12 it's clear that to obtain D, y varies between 0 and 2. Then for each fixed y-value we take the x values between the curve $y = \sqrt{x}$ (which is equivalent to $y^2 = x$) and 4. Therefore we see that D can also be described by the inequalities $0 \le y \le 2$ and $y^2 \le x \le 4$. Therefore we can iterate the double integral as

$$\begin{aligned}\iint_D f(x,y)dA &= \int_{y=0}^{2}\int_{x=y^2}^{4} \frac{y}{x^2+1}\,dx\,dy \\ &= \int_0^2 y\,[\arctan x]_{y^2}^4\,dy \\ &= \int_0^2 y\left(\arctan 4 - \arctan y^2\right) dy \\ &= \arctan 4\left[\frac{y^2}{2}\right]_0^2 - \int_0^2 y \arctan y^2\,dy.\end{aligned}$$

Now, to evaluate the remaining integral we first let $t = y^2$ so that $dt = 2y\,dy$. We get

$$\iint_D f(x,y)dA = 2\arctan 4 - \frac{1}{2}\int_0^4 \arctan t\,dt,$$

which we integrate by parts to obtain

$$= 2\arctan 4 - \frac{1}{2}\left([t\arctan t]_0^4 - \int_0^4 \frac{t}{1+t^2}dt\right).$$

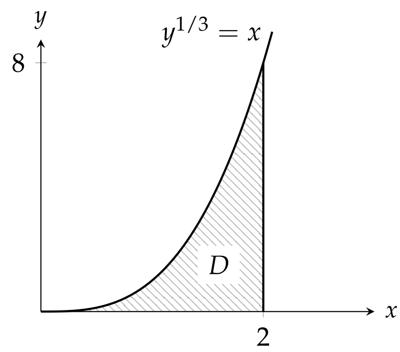


Figure 5.13: We have to parametrise the boundaries of the region D as functions of x in order to do the integration.

The first two factors cancel each other out, and the remaining integral is what we already evaluated before in (5.9), so we obtain

$$\int_0^2 \int_{y^2}^4 \frac{y}{x^2+1} dx\, dy = \frac{1}{4} \ln 17,$$

as required.

Next we'll look at an example that I promised to do before: an integral which is impossible to evaluate without changing the order of integration.

Example. Let

$$f(x, y) = e^{x^4}$$

on the domain D given by the inequalities $0 \leq y \leq 8$ and $y^{1/3} \leq x \leq 2$. We want to evaluate the iterated integral

$$\int_{y=0}^8 \int_{x=y^{1/3}}^2 e^{x^4} dx\, dy.$$

We discussed long time ago how functions of this type have no antiderivative in terms of elementary functions and we used Taylor series to integrate them instead. It would actually be possible to evaluate the value in this example by using Taylor series, but one can argue that the method I present here is much simpler. Anyway, by changing the order of integration we can introduce an extra factor of x^3, which saves the day. Let's do this. First we sketch the domain as follows, see Figure 5.13. Therefore, on D, x varies from 0 to 2. For each fixed x, we see that y varies between 0 and the curve $x = y^{1/3}$, that is, $x^3 = y$. Hence D is also described by the conditions $0 \leq x \leq 2$ and $0 \leq y \leq x^3$. So we can exchange the order of integration to get

$$\begin{aligned}\int_{y=0}^8 \int_{x=y^{1/3}}^2 e^{x^4} dx\, dy &= \int_{x=0}^2 \int_{y=0}^{x^3} e^{x^4} dy\, dx \\ &= \int_0^2 x^3 e^{x^4} dx.\end{aligned}$$

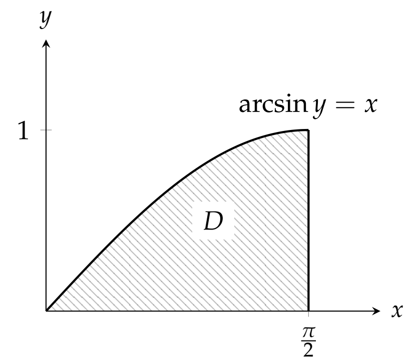


Figure 5.14: The region D corresponding to $0 \le y \le 1$ and $\arcsin y \le x \le \frac{\pi}{2}$.

Then we just let $u = x^4$ (so $du = 4x^3 dx$) and we get

$$\begin{aligned} &= \frac{1}{4}\int_0^{16} e^u du \\ &= \frac{1}{4}\left[e^u\right]_0^{16} = \frac{1}{4}\left(e^{16}-1\right). \end{aligned}$$

Here's another similar example.

Example. We want to evaluate

$$\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1+\cos^2 x}\, dx\, dy.$$

The domain of integration described by the limits is $0 \le y \le 1$ and $\arcsin y \le x \le \frac{\pi}{2}$, which we can see in Figure 5.14. In order to exchange the order of integration we notice that in the domain of integration x varies from 0 to $\frac{\pi}{2}$, and for each fixed x, y varies between 0 and the curve $x = \arcsin y$, that is, $\sin x = y$. Thus

$$\begin{aligned} \int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1+\cos^2 x}\, dx\, dy &= \int_0^{\pi/2} \int_0^{\sin x} \cos x \sqrt{1+\cos^2 x}\, dy\, dx \\ &= \int_0^{\pi/2} \sin x \cos x \sqrt{1+\cos^2 x}\, dx. \end{aligned}$$

We then do the substitution $u = 1 + \cos^2 x$, which satisfies $du = -2\sin x \cos x\, dx$. This turns the above integral into

$$-\frac{1}{2}\int_2^1 u^{1/2} du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_1^2 = \frac{1}{3}\left(2\sqrt{2}-1\right).$$

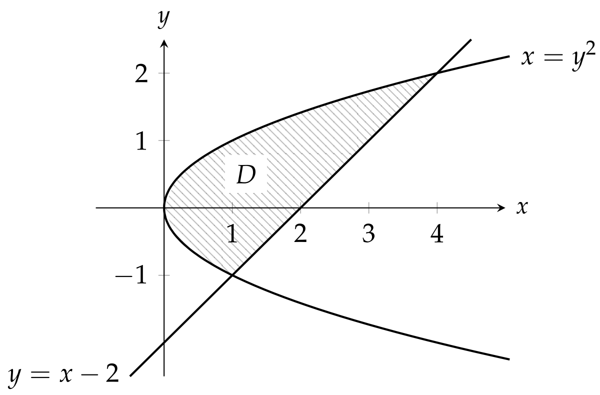


Figure 5.15: The region D bounded between the curves $y = x - 2$ and $x = y^2$.

Before moving on, I want to do one further example on iterated integrals.

Example. We want to evaluate

$$\iint_D y \, dA,$$

where D is the bounded region between the curves $y = x - 2$ and $x = y^2$. In order to iterate the integral, we have to write down D in terms of inequalities involving x and y. First we have to find where the curves cross. Substituting the first equation into the second gives $x = (x-2)^2$, which simplifies to $x^2 - 5x + 4 = 0$ and we can factor it as $(x-1)(x-4) = 0$. This gives the points of intersection as $(1, -1)$ and $(4, 2)$. We can now sketch the domain as in Figure 5.15. You could try to describe this region with either of x or y as the free variable. I choose to use y as free since then I don't have to deal with square roots. Then D can be described by the conditions $-1 \leq y \leq 2$ and $y^2 \leq x \leq y + 2$. Thus we can iterate our integral as

$$\begin{aligned}
\iint_D y \, dA &= \int_{y=-1}^{2} \int_{y^2}^{y+2} y \, dx \, dy \\
&= \int_{-1}^{2} y(y + 2 - y^2) dy \\
&= \left[\frac{y^3}{3} + y^2 - \frac{y^4}{4} \right]_{-1}^{2} \\
&= \frac{9}{4}.
\end{aligned}$$

If we wanted to try and iterate the other way around, then we have to break up the region D into two parts. This is because from Figure 5.15 you can see that on the left the domain is bounded from below by the curve $y^2 = x$, whereas on the right it's bounded from below by $y = x - 2$. Thus in the first part $0 \leq x \leq 1$ and $-\sqrt{x} \leq y \leq \sqrt{x}$, whereas in the second part $1 \leq x \leq 4$ and $x - 2 \leq y \leq \sqrt{x}$. Therefore our integral can also be iterated as

$$\iint_D y \, dA = \int_{x=0}^{1} \int_{y=-\sqrt{x}}^{\sqrt{x}} y \, dy \, dx + \int_{x=1}^{4} \int_{y=x-2}^{\sqrt{x}} y \, dy \, dx.$$

You could check yourself that we get the same answer as before.

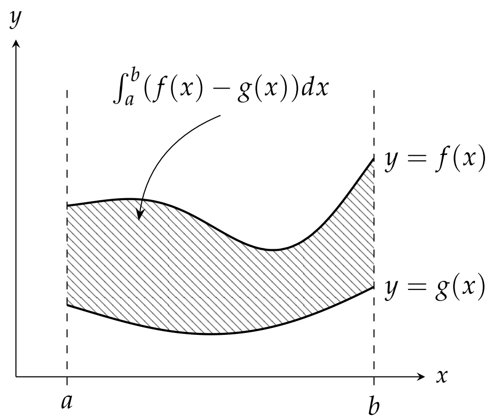


Figure 5.16: The area bounded by two curves is calculated as an integral of the difference of the functions.

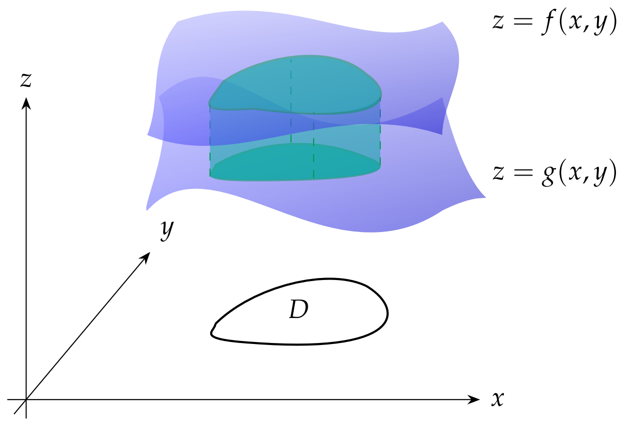


Figure 5.17: We want to find the volume of the solid bounded between two surfaces above a certain region D.

Our next goal is to compute volumes of solids bounded between two surfaces given as graphs of functions. The method is analogous to how we compute the area bounded between graphs of two functions of one variable. That is, consider two curves $y = f(x)$ and $y = g(x)$. To makes things easier, let's suppose both are positive functions and $f(x) \geq g(x)$. Then the area bounded between these two curves where x varies from a to b, is just $\int_a^b (f(x) - g(x))dx$, as we see in Figure 5.16. Similary, in three dimensions, we have two surfaces $z = f(x, y)$ and $z = g(x, y)$, and we want to find the volume of the solid described by the inequalities $g(x, y) \leq z \leq f(x, y)$, where (x, y) are restricted to some domain D. Then this volume (assuming z is always positive) is given by

$$\iint_D (f(x, y) - g(x, y))dA. \tag{5.10}$$

If there are parts of the surfaces that lie below the xy-plane, then we have to break up the domain D and consider those parts separately (since we are trying to compute the total *volume*, not the signed volume). This is depicted in Figure 5.17.

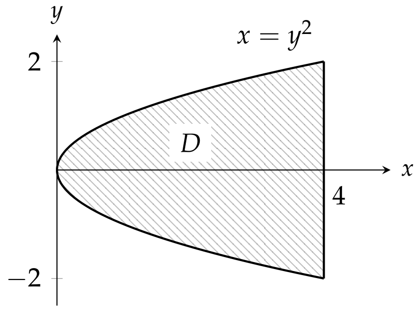


Figure 5.18: The domain D enclosed by the curves $x = y^2$ and $x = 4$.

Example. Find the volume V of the solid under the surface

$$z = 1 + x^2y^2,$$

and above the region D enclosed by the curves $x = y^2$ and $x = 4$. To find the limits of integration, we first sketch the region D. Clearly the curves intersect at $(4, 2)$ and $(4, -2)$. Hence we get the picture that you see in Figure 5.18. It follows that the domain D can be described by the inequalities $-2 \leq y \leq 2$, $y^2 \leq x \leq 4$. You could try the other way around as well, I just don't like square roots. To phrase this question as a volume between two surfaces, notice that we're just asking for the volume between $z = 1 + x^2y^2$ and $z = 0$. Thus by (5.10), we have that

$$\begin{aligned}
V &= \iint_D (1 + x^2y^2 - 0)dA \\
&= \int_{-2}^{2} \int_{y^2}^{4} \left(1 + x^2y^2\right) dx\, dy \\
&= \int_{-2}^{2} \left[x + \frac{x^3}{3}y^2\right]_{x=y^2}^{4} dy \\
&= \int_{-2}^{2} \left(4 + \frac{61}{3}y^2 - \frac{y^8}{3}\right) dy \\
&= \frac{2336}{27}.
\end{aligned}$$

You can see the solid graphed in Figure 5.19.

Next we'll do the "piece of cake" example.

Example. Find the volume bounded by the cylinder

$$x^2 + y^2 = 1,$$

and the planes $y = z$, $x = 0$ and $z = 0$ in the first octant. The first octant is the part of the three-space contained in $x \geq 0$, $y \geq 0$ and $z \geq 0$. This solid looks like an odd slice of (your favourite) cake, as we see in Figure 5.20. It seems someone has eaten all the tasty frosting already! We want to compute how much of the cake remains for you to eat. To iterate the integral we have to figure out the domain D of allowed x and y-values. To do this we project our restrictions to the xy-plane. The planes just give the boundary

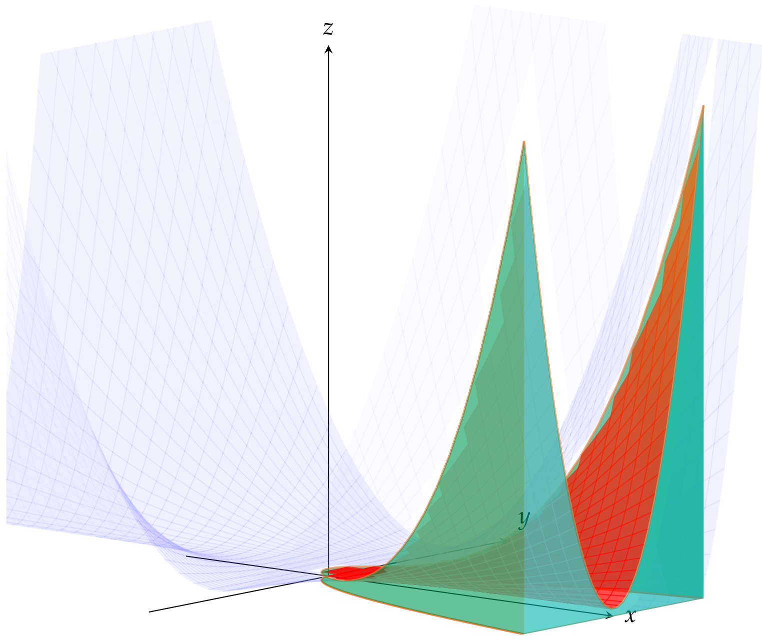


Figure 5.19: The solid bounded between the surfaces $z = 1 + x^2y^2$ and $z = 0$, above the region enclosed by the domain D.

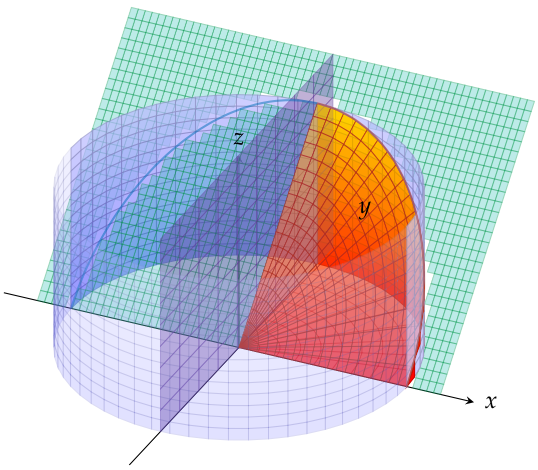


Figure 5.20: In this example the cake is not a lie.

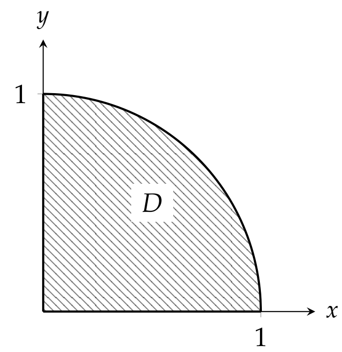


Figure 5.21: The domain of integration D is a quarter of a disk.

of the first quadrant on the xy-plane, so therefore we are considering points in the disk with boundary $x^2 + y^2 = 1$, see Figure 5.21. Later we'll see how to parametrise this region with polar coordinates, but for now we can notice that D is easily described by $0 \leq y \leq 1$ and $0 \leq x \leq \sqrt{1-y^2}$. The solid in question lies *below* the plane $y = z$. Hence its volume V is given by

$$\begin{aligned} V &= \iint_D y\, dA \\ &= \int_0^1 \int_0^{\sqrt{1-y^2}} y\, dx\, dy \\ &= \int_0^1 y\sqrt{1-y^2}\, dy. \end{aligned}$$

Now let $u = 1 - y^2$ and $du = -2y\, dy$. Thus

$$\begin{aligned} V &= \frac{-1}{2} \int_1^0 u^{1/2} du \\ &= \frac{1}{2} \left[\frac{2}{3} u^{3/2} \right]_0^1 \\ &= \frac{1}{3}. \end{aligned}$$

As a final comment I want to show you quickly how to express the above volume as a triple integral. This comment *won't be part of the examinable material*, but on the other hand there's not really anything new required as it is a straightforward extension of the ideas for double integrals. In essence, triple integrals are to volume as double integrals are to area. That is, if I have a solid S in three-space, then it's volume $V$is given by integrating 1 over it, that is, $V = \iiint_S 1\, dV$. If we take S to be the solid defined in the above example, then we can describe it in terms of the inequalities $0 \leq z \leq y$, $0 \leq y \leq 1$ and $0 \leq x \leq \sqrt{1-y^2}$. Notice that only the inequality involving z is new. We can then iterate the volume integral as

$$\begin{aligned} V &= \iiint_S 1\, dV \\ &= \int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^y 1\, dz\, dx\, dy. \end{aligned}$$

Now, if we evaluate the integral over z, we see that we get the same expression for the volume as we did in the above example!

$$V = \int_0^1 \int_0^{\sqrt{1-y^2}} y \, dx \, dy.$$

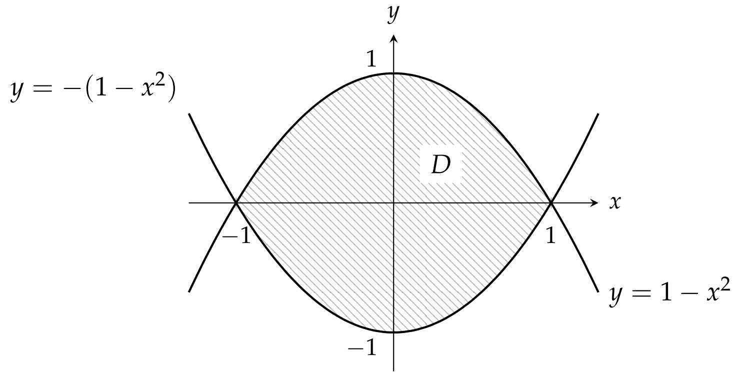


Figure 5.22: The region of integration is bounded between the two parabolas.

Here's one more example of volume integration that I meant to do in class today, but I thought I had spotted a mistake so I chose to delay it. I'll put it in today's notes anyway.

Example. Find the volume of the solid enclosed by the parabolic cylinders $y = 1 - x^2$ and $y = -(1 - x^2)$, and the planes $P_1 : x + y + z = 2$ and $P_2 : 2x + 2y - z + 10 = 0$.

In order to answer this question we have to first figure out what is our domain of integration D. In other words, what are allowed x and y-values. For this we look at the parabolic cylinders on the xy-plane. If we set them equal to each other, we can find the intersection points of the curves. We see that $1 - x^2 = -(1 - x^2) \iff 2(1 - x^2) = 0 \iff x = \pm 1$. This tells us that the curves intersect at $(1, 0)$ and $(-1, 0)$. Therefore we are integrating the surfaces over a domain D which satisfies e.g. $-1 \leq x \leq 1$ and $-(1 - x^2) \leq y \leq 1 - x^2$, as you can see in Figure 5.22. Next we need to figure out which of the surfaces is the upper one and which the lower one (above the region of integration[4]). Recall my comment that strictly speaking this is unnecessary since volume should always be positive so one can always choose the appropriate sign. On the other hand, this comment only makes sense if the lower surface never goes above the upper surface in the domain of integration, and if both surfaces are above the xy-plane. To check these things it's often just easier to first classify which surface lies above the other.

Let's make this concrete. In this question I want to determine which of the planes is above the other. The plane P_1 can be solved for z as $z = 2 - x - y$ and P_2 as $z = 2x + 2y + 10$. We abuse notation a bit and write these equations as $P_1(x, y) = 2 - x - y$ and $P_2(x, y) = 2x + 2y + 10$. Let's look at $P_1 - P_2$. If this quantity is positive then it means P_1 is above P_2, and if it's negative then P_1 lies below P_2 (make sure you understand why). We get $P_1 - P_2 = -3(x + y) - 8$. To determine whether the right-hand side is positive or negative we have to see what are the minimal and maximal values of $x + y$ in D. Clearly a very crude bound is $|x + y| \leq 2$ (since the domain D is certainly contained in the rectangle of width and height 2 centred at the origin). For us this crude bound turns out to be enough. It follows that $-6 - 8 \leq -3(x + y) - 8 \leq 6 - 8 \leq 0$. Therefore $P_1 - P_2 \leq 0$ and we can deduce that P_2 lies above P_1 throughout the domain D. It remains to check whether P_1 is non-negative (if it does then of course so does P_2). But since $|x + y| \leq 2$ it follows that certainly $2 - x - y \geq 0$. Now we can do the integration

[4]Since we have two intersecting planes which are not horizontal or vertical, then naturally there will be parts of the three-space where one plane lies above the other and parts where the roles are reversed.

correctly. It is important to spend some time and carry out these kind of calculations, especially in questions where the answer is not immediately "clear" (whatever I mean by that). The volume V of the solid is then given by

$$\begin{aligned} V &= \iint_D (P_2(x,y) - P_1(x,y))\, dA \\ &= \iint_D (2x + 2y + 10 - (2 - x - y))\, dA \\ &= \iint_D (3x + 3y + 8)\, dA. \end{aligned}$$

An attentive student could now notice that we can speed up our computations at this point! Indeed, the domain D is symmetric about the x and y-axes, so by the same argument as in the previous examples where we exploited symmetry, the integrals over x and y vanish. More precisely, e.g. $\iint_D x\, dA = 0$, because x is an odd function, and since D is symmetric about the y-axis it follows that for each point (x_0, y_0) on the right half of D there is a corresponding point $(-x_0, y_0)$ on the left half where the integrand takes an equal but opposite value (so this example is a bit simplified version of the arguments we've seen before). Similarly, $\iint_D y\, dA = 0$. Thus we can simplify and iterate our integral as

$$\begin{aligned} V &= \iint_D 8\, dA \\ &= 8 \int_{-1}^{1} \int_{-(1-x^2)}^{1-x^2} dy\, dx \\ &= 8 \int_{-1}^{1} 2(1 - x^2) dx \\ &= \frac{64}{3}. \end{aligned}$$

Here I parametrised the region D without further explanations, but by now you should be able to understand how I did it.

5.3 Double Integrals in Polar Coordinates

Recommended problems from §15.3: **13, 25, 31**.

One of the most important techniques for evaluating single variable integrals is integration by substitution. One might wonder how this works in two dimensions and whether it is equally useful. The answer is an emphatic *yes*, and even more so because the regions that we have to parametrise are more complicated than in one variable case (where your domain of integration is always an interval). I will show you how the change of variables works in the most important special case – *the polar coordinates*. I will also briefly allude into how one might go about doing an arbitrary change of variables, but for this course it is not examinable material.

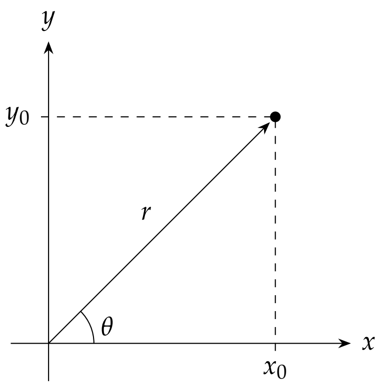


Figure 5.23: The definition of polar coordinates.

The polar coordinates are given by the relations

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$

If we take a fixed point (x_0, y_0) with polar coordinates r and θ, then it turns out that r gives the distance of the point from the origin, whereas θ is the angle between the positive x-axis and the line segment connecting the point to the origin (measured counter-clockwise). This looks like in Figure 5.23. Sometimes we write this point either as $(x, y)_{\text{rec}}$ or $(r, \theta)_{\text{polar}}$ to specify which coordinate system we are using (rec corresponds to rectangular). For any fixed point (x_0, y_0) we can always assume that $r > 0$ by choosing a suitable θ (this will be useful when computing e.g. limits). On the other hand when dealing with equations we'll see that sometimes it's practical to interpret negative r values as going in the opposite direction.

We can also try to write down r and θ in terms of x and y. For r this is easy. We immediately have

$$r^2 = x^2 + y^2. \tag{5.11}$$

On the other hand, for θ there is no single easy formula that works everywhere. If x and y are in the first quadrant (like in Figure 5.23) then clearly (for $x > 0$)

$$\theta = \arctan\frac{y}{x}.$$

On the other hand, by simple trigonometry you can see that in the other quadrants one has to resort to slight variations of the above formula. All of these formulas for θ combined are sometimes called the atan2-function.

The fundamental example in the use of polar coordinates is the unit circle. This can simply be expressed as the equation $r = 1$. We interpret equations like this by allowing θ to vary. In this case r is constant for all θ so it traces out the unit circle. If we want to describe the unit disk (so unit circle and its interior), it can be expressed by the inequalities $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Our ultimate goal is to use polar coordinates to evaluate double integrals. Denote the unit disk by D. We know that it has area π, so we can write

$$\iint_D 1\, dA = \pi.$$

We would like to arrive at this answer through the use of polar coordinates. The following derivation is *non-examinable*. That is, in the above integral we want to do the change of variables into polar coordinates. This corresponds to the coordinate transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{\varphi}{\longmapsto} \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix},$$

where I've denoted the transformation by φ. In single variable integration if you do a substitution $u = f(x)$, you have to take into account how the differentials transform as $du = f'(x)dx$. The way to generalise this into higher dimensions is through the determinant of the Jacobian matrix of φ. You can also think of the $f'(x)$ as a determinant of the Jacobian matrix of the transformation $x \overset{u}{\mapsto} f(x)$, since a 1×1 matrix is nothing more than just a number and its determinant is itself! Anyway, for our φ its Jacobian is of course

$$\varphi' = \begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

If we compute the determinant of this matrix we get

$$\det\varphi' = r\cos^2\theta + r\sin^2\theta = r.$$

It turns out that the determinant of the Jacobian matrix measures how area changes under the transformation φ. That is if we take a small rectangle R of known area, then what is the area of $\varphi(R)$? In rectangular (cartesian) coordinates the area element dA is just given by $dx\,dy$. So we can think of dA as the area of a rectangle with side lengths dx and dy. If we then apply φ to this rectangle, we would like to express the area in terms of dr and $d\theta$. As stated above, it turns out that the factor we need to correct this change in area is exactly the determinant of the Jacobian, so we can write

$$dx\,dy = r\,dr\,d\theta.$$

I won't go into more detail about why this is the case in general. Instead I'll demonstrate it in the case of polar coordinates. Suffices to say that to do an arbitrary coordinate transformation φ, you proceed in exactly the same steps and find the determinant of the Jacobian of φ (provided its partial derivatives satisfy some smoothness conditions), and then apply the transformation to the integrand and to the domain of integration.

Let's see what the above argument looks like geometrically. We want to take a small "box" in the r & θ coordinates and compute its area. That is, for a fixed point $(r, \theta)_{\text{polar}}$ we want to find the area of the box with the opposite corner at $(r + dr, \theta + d\theta)_{\text{polar}}$. We can sketch this as in Figure 5.24. So we see from the picture that we have the same expression that we got before for the area element dA in polar coordinates:

$$\boxed{dA = r\,dr\,d\theta.}$$

Now, if we want to iterate an integral in polar coordinates, we just have to remember to use the above expression to substitute for dA. To go back to our example of finding the

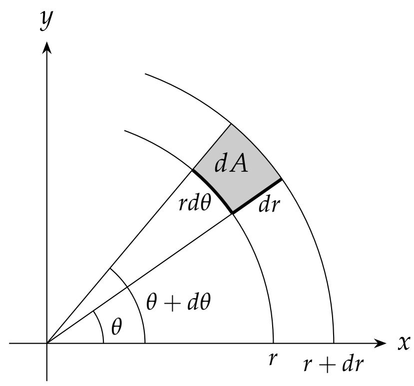


Figure 5.24: The area element in polar coordinates is $r\,dr\,d\theta$.

area of the unit disk, we see that in polar coordinates the integral becomes

$$\begin{aligned}\iint_D 1\,dA &= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} r\,d\theta\,dr\\ &= \int_0^{2\pi} d\theta \int_0^1 r\,dr\\ &= [\theta]_0^{2\pi}\left[\frac{r^2}{2}\right]_0^1\\ &= \pi,\end{aligned}$$

which is what we wanted to show. Now let's do a more complicated example.

Example. Find the volume of the solid bounded by the paraboloid $z = 1 + 2x^2 + 2y^2$ and the plane $z = 7$ in the first octant (where $x, y, z \geq 0$).

The first thing to do (as usual) is to figure out the domain of integration D. In this example it's not quite as straightforward as in the previous ones, although it's still not difficult. We have to figure out what part of the three space is given by the conditions and then look at its projection to the xy-plane. In less technical terms, we just have to find all the x and y values above which there is something to integrate over (i.e. parts of our solid). It is helpful to plot the surfaces. We do this in Figure 5.25. It is then clear that the domain of integration can be found by considering the curve of intersection of the two surfaces and projecting it to xy-plane. We get $7 = 1 + 2x^2 + 2y^2$, which gives $x^2 + y^2 = 3$, which is the equation of a circle of radius $\sqrt{3}$, centred at the origin. We're restricted to the first octant so therefore we only take the first quarter of this disk, as shown in Figure 5.25. We can parametrise this disk in polar coordinates by the

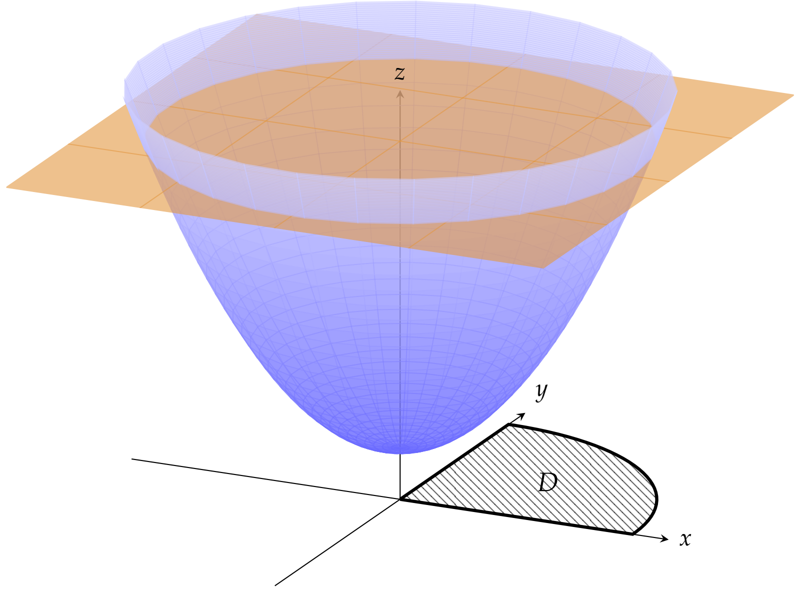


Figure 5.25: The paraboloid $z = 1 + 2x^2 + 2y^2$ cut off by the plane $z = 7$ and the corresponding domain of integration on the xy-plane.

conditions $0 \le r \le \sqrt{3}$ and $0 \le \theta \le \frac{\pi}{2}$. Therefore we can iterate to find the volume as

$$\begin{aligned} V &= \iint_D (7 - (1 + 2x^2 + 2y^2))\, dA \\ &= \int_0^{\sqrt{3}} \int_0^{\pi/2} (6 - 2r^2) r \, d\theta \, dr \\ &= \int_0^{\sqrt{3}} (6r - 2r^3) dr \int_0^{\pi/2} d\theta \\ &= \left[3r^2 - \frac{r^4}{2} \right]_0^{\sqrt{3}} \frac{\pi}{2} \\ &= \frac{9\pi}{4}. \end{aligned}$$

Next we'll study how to read equations in polar coordinates. Just like in the rectangular coordinates we usually specify y as a function of x, in polar coordinates we usually specify r as a function θ, i.e. $r = f(\theta)$. A good strategy to plot equations like this is to investigate various values of θ, in particular any values of θ for which r becomes 0.

Example. We already saw how the equation $r = 1$ gives the unit circle. This is a special case of equations of the type $r = \cos(a\theta)$ for $a = 0$. Let's look at $a = 1$, that is, $r = \cos\theta$. For $\theta = 0$ we of course get $r = 1$. So our initial point along the x-axis is at $(1, 0)$. Now, the first time $\cos\theta$ is equal to 0 (when we increase θ from 0) is when $\theta = \frac{\pi}{2}$. Since $\cos\theta$ decreases from 1 to 0 on $[0, \frac{\pi}{2}]$, we can see that we get a semicircle. But then for values of θ beyond $\frac{\pi}{2}$, $\cos\theta$ will be negative, e.g. $\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$. Therefore the points corresponding to these θ values will lie to the *right* of the y-axis (recall our interpretation for negative r-values). In particular $\cos\pi = -1$ so we get back to where we started, and we've traced out a circle of radius $1/2$ centred at $(1/2, 0)$, see Figure 5.26. If we consider $\theta \in [\pi, 2\pi]$, then we just trace out the same circle again.

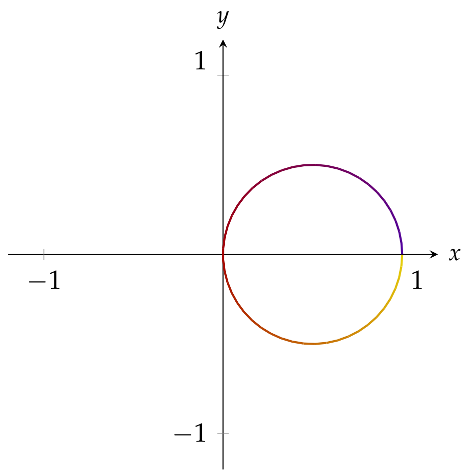


Figure 5.26: Plot of the equation $r = \cos\theta$ for $\theta \in [0, \pi]$. Change in the shade corresponds to increasing θ (purple is $\theta = 0$, yellow is the upper bound on θ).

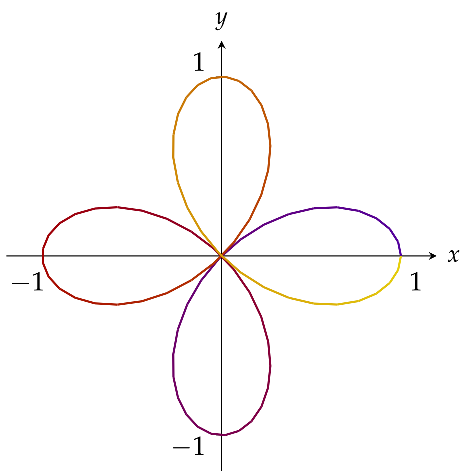


Figure 5.27: Plot of the equation $r = \cos 2\theta$ for $\theta \in [0, 2\pi]$.

Example. Let's now look at the case $a = 2$, that is, the equation

$$r = \cos 2\theta.$$

Again, at $\theta = 0$ we have that $r = 1$. Next, let's find the first value of θ when $r = 0$. This happens when $2\theta = \frac{\pi}{2}$, which gives $\theta = \frac{\pi}{4}$. Similarly $\theta = -\frac{\pi}{4}$ gives $r = 0$. Since $\cos 2\theta$ is a decreasing function when θ starts from 0, we see that the range $\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$ traces out a kind of a leaf of a "clover". By a similar computation, the next leaf of the clover corresponds to $\theta \in [\frac{\pi}{4}, \frac{3\pi}{4}]$, which occupies the lower y-axis, and so on for further values of θ. Make sure you check yourself that this is the case. Eventually we're lead to the picture seen in Figure 5.27.

Example. Let's now take the case $a = 3$. We can work as before to deduce that the first leaf of this graph is given by the range $[-\frac{\pi}{6}, \frac{\pi}{6}]$ and are led to the picture in Figure 5.28. We want to find out the area enclosed in the first leaf. Let's denote this region by D. By the above observations we can parametrise D by letting $0 \leq r \leq \cos 3\theta$ and $-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$.

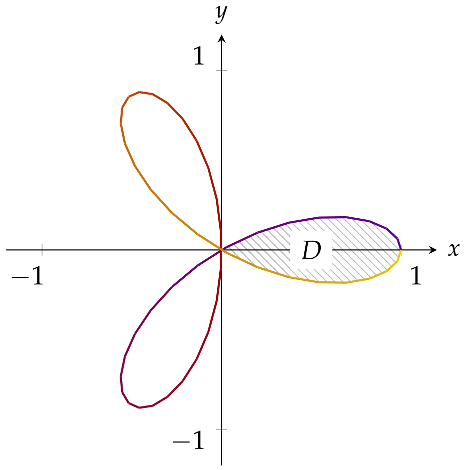


Figure 5.28: Plot of the equation $r = \cos 3\theta$ for $\theta \in [0, \pi]$.

We find the area by integrating 1 over D:

$$\begin{aligned}\text{area}(D) &= \iint_D 1\, dA \\ &= \int_{-\pi/6}^{\pi/6} \int_0^{\cos 3\theta} r\, dr\, d\theta \\ &= \tfrac{1}{2} \int_{-\pi/6}^{\pi/6} \tfrac{1}{2} \cos^2 3\theta\, d\theta.\end{aligned}$$

To do this integration we use the double angle formula $\cos 2\alpha = 2\cos^2 \alpha - 1$ for $\alpha = 3\theta$. Thus

$$\begin{aligned}\text{area}(D) &= \tfrac{1}{4} \int_{-\pi/6}^{\pi/6} (\cos 6\theta + 1)\, d\theta \\ &= \tfrac{1}{4} \left[\tfrac{1}{6} \sin 6\theta + \theta\right]_{-\pi/6}^{\pi/6} \\ &= \frac{\pi}{12}.\end{aligned}$$

In myCourses you can find an animation displaying the graph of $r = \cos a\theta$ for a ranging from 0 to 6 (plotted for $\theta \in [0, 2\pi]$). Observe that when a is an even integer the graph is symmetric about both the x and y-axes. On the other hand, if a is an odd integer then there is symmetry only about the x-axis. Moreover, for odd a you can notice that the curve is traced twice when θ varies from 0 to 2π!

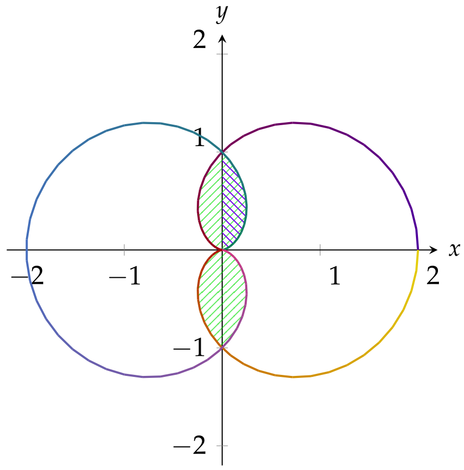


Figure 5.29: Plot of the cardioids $r = 1 + \cos\theta$ (in purple/yellow) and $r = 1 - \cos\theta$ (in green/magenta).

Here are a few more examples of finding areas in polar coordinates (both of them are exercises from the book, as many of our examples are).

Example. Find the area enclosed between both of the cardioids

$$r = 1 + \cos\theta, \qquad\qquad r = 1 - \cos\theta.$$

First it's a good idea to figure out what the area looks like. We can plot the cardioids by using the same technique as last time. In the first equation for $\theta = 0$ we get $r = 2$ so we start at the $(2, 0)$ point on the xy-plane. The first time r will be zero for the first equation is when $\cos\theta = -1$, that is $\theta = \pm\pi$. It's also useful to check that for $\theta = \pm\frac{\pi}{2}$ we obtain $r = 1$ in the first equation. This should be enough to figure out that the cardioid looks like in Figure 5.29, where we've also plotted the second equation by using similar techniques and highlighted the area enclosed by the cardioids. In particular, the part to the right of the y-axis in the second equation corresponds to $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. By symmetry, the total shaded area is equal to 4 times the area enclosed by the second cardioid in the first quadrant (you can also see this in Figure 5.29 as the cross-hatched area). This region, denote it by D_1, is then parametrised by $0 \le \theta \le \frac{\pi}{2}$ and $0 \le r \le 1 - \cos\theta$ (since we want all the points *between* the origin and the curve $r = 1 - \cos\theta$, in this particular region). Thus the area of the total shaded region, D, is given by

$$\begin{aligned}
\text{area}(D) &= 4 \iint_{D_1} dA \\
&= 4 \int_0^{\pi/2} \int_0^{1-\cos\theta} r \, dr \, d\theta \\
&= 4 \int_0^{\pi/2} \frac{1}{2}(1 - \cos\theta)^2 d\theta \\
&= 2 \int_0^{\pi/2} \left(\frac{3}{2} + \frac{1}{2}\cos 2\theta - 2\cos\theta \right) d\theta,
\end{aligned}$$

by the double angle formula $\cos 2\theta = 2\cos^2\theta - 1$. We can then easily integrate to find

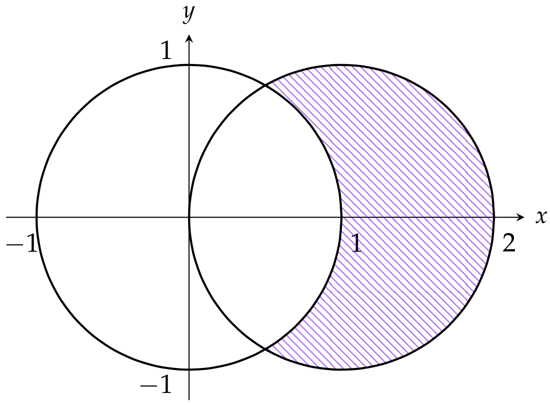


Figure 5.30: The area inside the circle $(x-1)^2+y^2=1$ and outside the circle $x^2+y^2=1$.

that

$$\begin{aligned}\operatorname{area}(D) &= 2\left[\frac{3}{2}\theta+\frac{1}{4}\sin 2\theta-2\sin\theta\right]_0^{\pi/2}\\ &= \frac{3\pi}{2}-4.\end{aligned}$$

Example. Find the area inside the circle $(x-1)^2+y^2=1$ and outside the circle $x^2+y^2=1$. We can plot the required region D as in Figure 5.30. We want to parametrise D in polar coordinates. To do this we first have to convert our equations into polar form. This is easily done just by substituting the definition of x and y in terms of r and θ. We get

$$\begin{aligned}(x-1)^2+y^2&=1\\ (r\cos\theta-1)^2+r^2\sin^2\theta&=1\\ r^2(\cos^2\theta+\sin^2\theta)-2r\cos\theta+1&=1\\ r^2&=2r\cos\theta,\end{aligned}$$

which gives $r=2\cos\theta$ or $r=0$. The other equation is easier since we know that the unit circle is just $r=1$. Now, we need to find where the curves intersect. To do this we substitute $r=1$ into the first equation and get $1=2\cos\theta$. This of course has solutions $\theta=\pm\frac{\pi}{3}$. It follows that D is given by $-\frac{\pi}{3}\le\theta\le\frac{\pi}{3}$ and $1\le r\le 2\cos\theta$.

If you are unsure about this here's another explanation: let's write the two curves as $r=f_1(\theta)$ and $r=f_2(\theta)$, where $f_1(\theta)=2\cos\theta$ and $f_2(\theta)=1$. Now, points on the *inside* of the first circle (we include the boundary) must satisfy $r\le f_1(\theta)$, whereas points on the *outside* of the second circle must satisfy $r\ge f_2(\theta)$. Make sure you understand why this is so (also in terms of the geometric meaning of r and θ). We need both of these conditions at once, i.e. $f_2(\theta)\le r\le f_1(\theta)$, which simplifies to $1\le r\le 2\cos\theta$. We then have to figure out for which values of θ this condition makes sense, that is, when is $2\cos\theta\ge 1$ (otherwise the inequality wouldn't make sense, so there would be nothing to integrate!). This condition rearranges to $\cos\theta\ge\frac{1}{2}$, which clearly happens for $\theta\in[-\frac{\pi}{3},\frac{\pi}{3}]$ (by inspecting the graph of $\cos\theta$, for example).

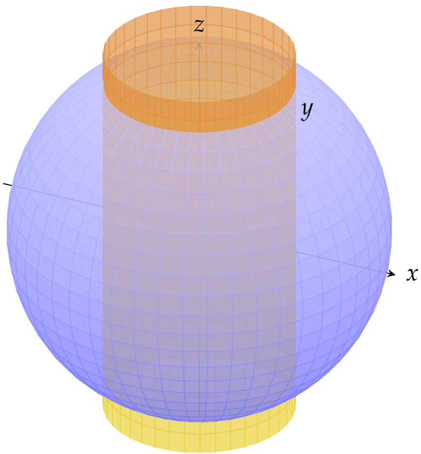


Figure 5.31: The volume inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$.

In any case, we're now ready to integrate.

$$\begin{aligned} A &= \int_{-\pi/3}^{\pi/3} \int_{1}^{2\cos\theta} r\,dr\,d\theta \\ &= \int_{-\pi/3}^{\pi/3} \frac{1}{2} \left(4\cos^2\theta - 1\right) d\theta \\ &= \frac{1}{2} \int_{-\pi/3}^{\pi/3} \left(2(\cos 2\theta + 1) - 1\right) d\theta \\ &= \left[\frac{1}{2}\sin 2\theta + \frac{1}{2}\theta\right]_{-\pi/3}^{\pi/3} \\ &= \frac{\sqrt{3}}{2} + \frac{\pi}{3}. \end{aligned}$$

Next, let's find some volumes with polar coordinates.

Example. Find the volume V inside the sphere

$$x^2 + y^2 + z^2 = 16,$$

and outside the cylinder $x^2 + y^2 = 4$. This situation looks like a deseeded apple, as we see in Figure 5.31. To simplify our computations a little bit, we just consider the part of the solid lying in the first octant. Denote the volume of this part by V_1. By symmetry there are 8 equal such parts in V so we get $V = 8V_1$. In this region we can write the equations of our surfaces as $r^2 = 4$ and $r^2 + z^2 = 16$. These simplify to $r = 2$ and $z = \sqrt{16 - r^2}$ (since z is positive in V_1). We then want to integrate over $0 \le \theta \le \frac{\pi}{2}$ (since we are in the first octant) and all r such that $r \ge 2$ (since we want to be outside the cylinder) and $r \le 4$ (since this is the largest value of r on the sphere). That is, if we denote our domain of integration by D_1 then it can be described as $0 \le \theta \le \frac{\pi}{2}$ and $2 \le r \le 4$. The upper surface of integration is then $z = \sqrt{16 - r^2}$ and the lower is $z = 0$.

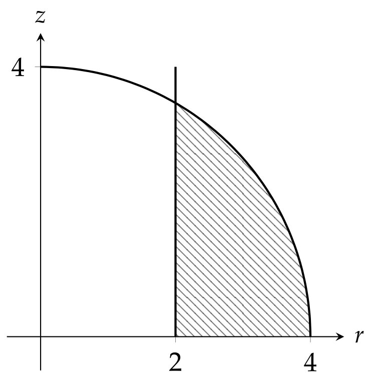


Figure 5.32: The surfaces $z = \sqrt{16-r^2}$ and $r = 2$ visualised from the side on the rz-plane with our solid highlighted.

Therefore we can iterate as

$$\begin{aligned} V &= 8V_1 \\ &= 8\iint_{D_1} \sqrt{16-r^2}dA \\ &= 8\int_2^4 \int_0^{\pi/2} \sqrt{16-r^2}r\,d\theta\,dr \\ &= 8\frac{\pi}{2}\left(\frac{-1}{2}\right)\int_{12}^0 u^{1/2}du, \end{aligned}$$

after doing the substitution $u = 16 - r^2$. Therefore

$$\begin{aligned} V &= 2\pi\left[\frac{2}{3}u^{3/2}\right]_0^{12} \\ &= \frac{4\pi}{3}(12)^{3/2} \\ &= 32\pi\sqrt{3}. \end{aligned}$$

Sometimes it can also be helpful to visualise situations like this on the rz-plane. That is, we plot the surfaces expressed in terms of r and z and then think of the solid as obtained by rotating the resulting picture about the z-axis. In our case (since we are restricting to the first octant) we would get what you see in Figure 5.32.

We finish off this lecture with a classical example.

Example. We want to prove that

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}.$$

This seems incredible, right? This is called the *Gaussian integral* and it appears in many places, for example, in the cumulative distribution function of the normal distribution (after a normalisation to make the total probability 1).

We use a couple of ingenious tricks to deduce this. Denote the integral by I. Then

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right),$$

where I've just used a different dummy variable y in the second integration (make sure you understand why it doesn't matter what I call the variable; in particular, if I want to combine the integrals I cannot use the same dummy variable x). As long as the integrals converge absolutely (I won't prove this, but it's not difficult) we can use Fubini's theorem to write I^2 as an iterated integral over $\mathbb{R}^2$ (so we're kind of working backwards compared to what we usually do, this is partly why this is so unintuitive). We get

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\, dy.$$

Clearly this step would make no sense if both integrals were written in terms of x. Now we want to parametrise the domain of integration $\mathbb{R}^2$ in polar coordinates. This is given by $0 \le r < \infty$, $0 \le \theta < 2\pi$. Notice that this parametrisation covers each point exactly once. Therefore we can transform I^2 into polar coordinates as

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r\, dr\, d\theta,$$

where we have to be careful not to forget the determinant of the Jacobian, r. But now the above integral is easy to evaluate, we let $u = r^2$ so that $du = 2r\, dr$ and

$$\begin{aligned} I^2 &= 2\pi \frac{1}{2} \int_0^{\infty} e^{-u} du \\ &= \pi \left[-e^{-u} \right]_0^{\infty} \\ &= \pi \left(1 - \lim_{u \to \infty} e^{-u} \right) \\ &= \pi. \end{aligned}$$

Therefore by taking square roots we obtain $I = \sqrt{\pi}$, which is what we wanted to show.

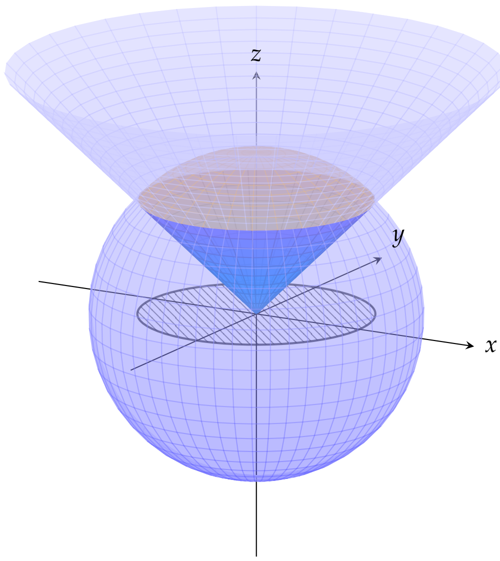


Figure 5.33: We're computing the volume of the ice cream cone as seen in the picture.

Before we go to the material I presented during today's lecture. I wanted to add one example on finding volumes that I haven't had time to do during the lectures.

Example. Find the volume V above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 1$. The situation is like in Figure 5.33, that is, we are looking at an ice cream cone and we want to compute how much of delicious (stracciatella) ice cream there is for you to eat. To find the domain of integration we want to see for which values of (x, y) is the cone *below* the sphere (why? look at the picture). Since we're on the upper half-space ($z \geq 0$, why?) we can solve the equation of the sphere for z as $z = \sqrt{1-x^2-y^2}$. We're then interested in finding x and y for which

$$\sqrt{x^2+y^2} \leq \sqrt{1-x^2-y^2}.$$

We can square this and rearrange to get

$$2x^2+2y^2 \leq 1,$$

that is

$$x^2+y^2 \leq \frac{1}{\sqrt{2}}.$$

This is our domain D. We can express this in polar coordinates as $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 1/\sqrt{2}$. Of course you could've immediately converted the equations of our surfaces into polar coordinates and found the domain that way as well (try it!). The cone becomes just $z = r$ and the sphere gives $z = \sqrt{1-r^2}$ in polar coordinates. If you like, you can also plot this situation on the rz-plane, which you can recall can be visualised as looking at the surfaces from the side and then rotating the picture around the z axis by a full 2π, we plot this in Figure 5.34. We can then express our volume integral and

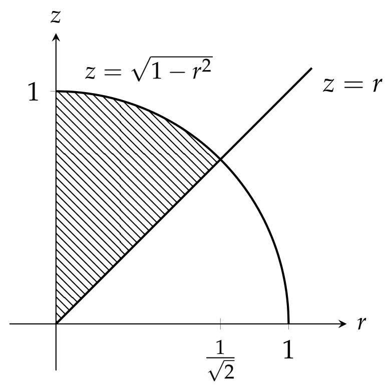


Figure 5.34: A view of the situation in the rz-axes. Think of this picture as revolved around the z axis to obtain our full solid.

iterate as

$$\begin{aligned} V &= \iint_D \left(\sqrt{1-x^2-y^2} - \sqrt{x^2+y^2} \right) dA \\ &= \int_0^{2\pi} \int_0^{1/\sqrt{2}} \left(\sqrt{1-r^2} - r \right) r\,dr\,d\theta \\ &= 2\pi \left(\int_0^{1/\sqrt{2}} r\sqrt{1-r^2}dr - \left[\frac{r^3}{3} \right]_0^{1/\sqrt{2}} \right). \end{aligned}$$

We let $u = 1 - r^2$ in the remaining integral which gives $du = -2r\,dr$. Thus

$$\begin{aligned} V &= 2\pi \left(\frac{-1}{2} \int_1^{1/2} u^{1/2} du - \frac{1}{6\sqrt{2}} \right) \\ &= 2\pi \left(\frac{1}{2} \left[\frac{2}{3} u^{3/2} \right]_{1/2}^{1} - \frac{1}{6\sqrt{2}} \right) \\ &= \frac{2\pi}{3} \left(1 - \frac{1}{\sqrt{2}} \right) \\ &= \frac{\pi}{3} \left(2 - \sqrt{2} \right). \end{aligned}$$

5.5 Area of Surfaces

Recommended problems from §15.5: **5, 7, 13**.

In this last section of the course we learn how to compute the area of a surface S given as a graph of a function, $z = f(x, y)$, lying above some region D on the xy-plane.

The way we approach this is the following idea: when we wanted to compute an area of a planar region (so a domain on the xy-plane) we just computed the integral of 1 with respect to the “flat” area element dA. We would like to have a similar quantity for

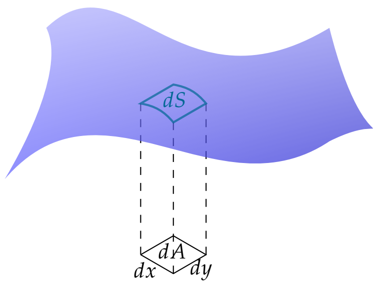


Figure 5.35: The relationship between dS and dA can be observed geometrically.

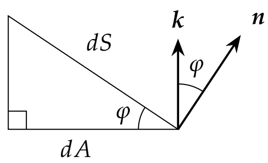


Figure 5.36: A simplified picture to show how to get dS in terms of dA.

the surface S, that is, we want a "curved" area element dS that lives on the surface such that if we integrate 1 with respect to dS we then get the required area. But what is dS? Recall that we can think of dA as a rectangle with sides dx and dy. Then dS is nothing more than dA lifted from the xy-plane to the surface dS (or in otherwords dA is the projection of dS to the xy-plane), see Figure 5.35. So now that we know what's going on, how do we relate dA to dS more precisely? This is simple with some trigonometry. Let's look at a simplified picture first. In Figure 5.36 we can think of looking at the situation of Figure 5.35 directly from the side, so the hypotenuse would correspond to dS and the bottom of the triangle to dA. Clearly then $dA = \cos\varphi\, dS$. But what is φ? It's an easy exercise to check that this is equal to the angle between a normal vector $\boldsymbol{n}$ to S and the vector $\boldsymbol{k}$ pointing directly up. But we know how to compute normal vectors to surfaces! This is nothing more than $\nabla(z - f(x,y))$[5]. Thus we get the more precise picture seen in Figure 5.37. Thus we can compute

[5]The normal vector to this surfaces is NOT $\nabla f(x,y)$, make sure you understand the difference. $\nabla f(x,y)$ is normal to the *level curves* of f, which are curves on the xy-plane.

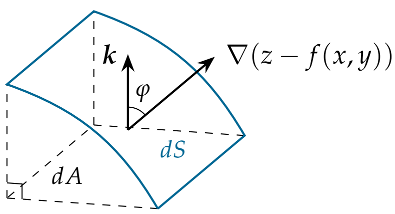


Figure 5.37: A more precise picture of the relationship between dS and dA.

$$\boldsymbol{n} = \nabla(z - f(x,y)) = \begin{pmatrix} -f_x(x,y) \\ -f_y(x,y) \\ 1 \end{pmatrix}.$$

By definition of the vector product we then have

$$\cos\varphi = \frac{\boldsymbol{n} \cdot \boldsymbol{k}}{|\boldsymbol{n}||\boldsymbol{k}|} = \frac{1}{\sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}}.$$

Thus we can substitute this into $dA = \cos\varphi \, dS$ to get

$$dS = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \, dA.$$

Since $z = f(x,y)$ we can write the partials also in terms of z to simplify the notation a bit. Therefore we arrive at our final formula

$$\boxed{\text{area of } S \text{ above } D = \iint_D 1 \, dS = \iint_D \sqrt{1 + \left(\tfrac{\partial z}{\partial x}\right)^2 + \left(\tfrac{\partial z}{\partial y}\right)^2} \, dA.} \tag{5.12}$$

Let's use this formula in some examples.

Example. Find the area of the part of the plane

$$3x + 2y + z = 6$$

that lies in the first octant (so $x, y, z \geq 0$). What is the domain of integration D? We have to see what is the intersection of our plane $z = 6 - 3x - 2y$ with the xy-plane (so $z = 0$). The solid for which we're computing the volume will look like a pyramid. This usually happens when you're looking at domains bounded by four planes (what are the 3 other planes in this question?). The curve of intersection is $3x + 2y = 6$, which gives $y = 3 - \frac{3}{2}x$. Thus we find that D is the area shown in Figure 5.38. There you can also see a depiction of the volume we're trying to find. Therefore D can be expressed as, for example, $0 \leq x \leq 2$ and $0 \leq y \leq 3 - \frac{3}{2}x$. Let's quickly verify that the part of the plane is actually above D. We had $z = 6 - 3x - 2y$, but on D we have that $y \leq 3 - \frac{3}{2}x$ (why?) so rearranging we get $2y + 3x \leq 6$, on the other hand both x and y are positive so we get $|2y + 3x| \leq 6$. Therefore it follows that $6 - 3x - 2y \geq 0$ in our domain of integration, as we wanted.

In order to use formula (5.12) we need to compute the partial derivatives of z. Of course $\frac{\partial z}{\partial x} = -3$ and $\frac{\partial z}{\partial y} = -2$. So

$$dS = \sqrt{1 + (-3)^2 + (-2)^2} \, dA = \sqrt{14} \, dA.$$

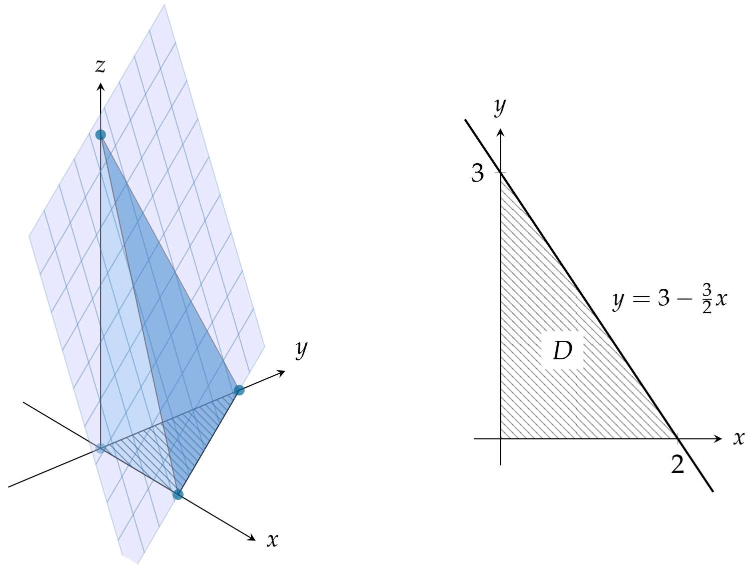


Figure 5.38: The intersection of four planes is a pyramid.

Therefore we can find the area of S above D as

$$\begin{aligned}\iint_D 1\,dS &= \iint_D \sqrt{14}\,dA \\ &= \sqrt{14}\int_0^2\int_0^{3-\frac{3}{2}x} dy\,dx \\ &= \sqrt{14}\left[3x - \frac{3}{4}x^2\right]_0^2 \\ &= \sqrt{14}(6-3) \\ &= 3\sqrt{14}.\end{aligned}$$

A nice shortcut here would be to notice that the integral with dA is nothing more than $\sqrt{14}$ times the area of D. Since D is a triangle this is easy to compute and you can verify that we get the same answer that way, too.

Example. We know from high-school geometry that the area of a sphere of radius a is $4\pi a^2$. Let's prove this.

We'll compute the area of a hemisphere of radius a (since we want our surface to be above the xy-plane) and multiply the result by two. Thus our surface is $x^2 + y^2 + z^2 = a^2$ for $z \geq 0$, which can be solved for $z = \sqrt{a^2 - x^2 - y^2}$. In polar coordinates this becomes $z = \sqrt{a^2 - r^2}$. In particular, we see that our domain of integration is when $\sqrt{a^2 - r^2} \geq 0$, which gives $a^2 \geq r^2 \geq 0$, or, $0 \leq r \leq a$ and naturally $0 \leq \theta \leq 2\pi$. For the partial derivatives it might be tempting to compute them in terms of the polar coordinates. But you can't just directly replace the partial derivatives w.r.t. x and y with those for r and θ.

In order to convert the formula properly you'd have to use the chain rule. Therefore, it's probably better just to not confuse yourself and start the computations in terms of x and y and convert to polar coordinates at the end. Therefore,

$$\frac{\partial z}{\partial x} = \frac{-2x\left(\frac{1}{2}\right)}{\sqrt{a^2 - x^2 - y^2}}, \qquad \frac{\partial z}{\partial y} = \frac{-2y\left(\frac{1}{2}\right)}{\sqrt{a^2 - x^2 - y^2}}.$$

We can substitute these into the formula for dS to get

$$\begin{aligned} dS &= \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}}\, dA \\ &= \sqrt{\frac{a^2 - x^2 - y^2 + x^2 + y^2}{a^2 - x^2 - y^2}}\, dA \\ &= \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dA. \end{aligned}$$

Now we can convert to polar coordinates since we know that $dA = r\, dr\, d\theta$, which gives

$$dS = \frac{a}{\sqrt{a^2 - r^2}} r\, dr\, d\theta.$$

We're then ready to compute the are of the hemisphere:

$$\begin{aligned} \text{area}(S) &= \iint_D 1\, dS \\ &= \int_0^{2\pi} \int_0^a \frac{ar}{\sqrt{a^2 - r^2}}\, dr\, d\theta \\ &= 2\pi \left(\frac{-1}{2}\right) \int_{a^2}^0 \frac{a\, du}{u^{1/2}}, \end{aligned}$$

by setting $a^2 - r^2 = u$. Thus

$$\begin{aligned} \text{area}(S) &= \pi a \left[2u^{1/2}\right]_0^{a^2} \\ &= 2\pi a^2. \end{aligned}$$

The are of the whole sphere is then twice the above, i.e. $4\pi a^2$.

Example. Find the area of the sphere

$$x^2 + y^2 + z^2 = 4z,$$

that lies inside the paraboloid $z = x^2 + y^2$. To graph this, it's useful to first transform the equation of the sphere into the standard form. Move $4z$ to the other side and complete the square and you get

$$x^2 + y^2 + (z - 2)^2 = 4.$$

Then we see that this solid is a bit like a filled round cup, as in Figure 5.39. We want the volume that is bounded above by the sphere and from below by the paraboloid. Last time I kept everything in x and y and found the domain of integration and then converted to polars so let's do it the other way around now (either works). The sphere

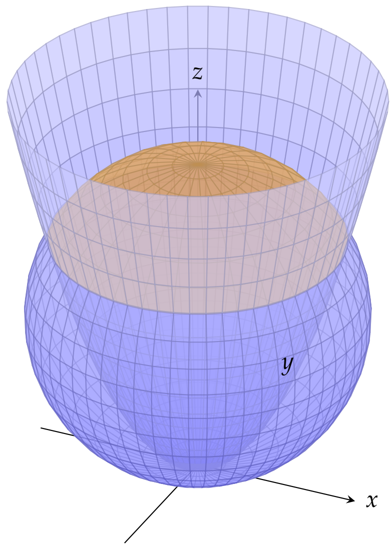


Figure 5.39: We want to find the area of the sphere above the paraboloid.

becomes $(z-2)^2 = 4 - r^2$, which we can solve (since we want the upper hemisphere) for z as $z = 2 + \sqrt{4-r^2}$. The paraboloid is of course $z = r^2$. The domain D is then the region where $2 + \sqrt{4-r^2} \geq r^2$. To solve this let's look at when is $2 + \sqrt{4-r^2} = r^2$. We move 2 to the other side and square to get

$$\begin{aligned} 4 - r^2 &= (r^2-2)^2 \\ 4 - r^2 &= r^4 - 4r^2 + 4 \\ r^4 - 3r^2 &= 0 \\ r^2(r^2-3) &= 0. \end{aligned}$$

Thus we get that $r = 0$ or $r = \sqrt{3}$ (since $r \geq 0$). You can verify (by e.g. plotting both of the curves on the rz-plane) that in this region $r^2 \leq 2 + \sqrt{4-r^2}$. Therefore our D is given by $0 \leq r \leq \sqrt{3}$ and $0 \leq \theta \leq 2\pi$. We already computed dS for the sphere in the previous example. In this case the radius is 2 so we get

$$dS = \frac{2r\,dr\,d\theta}{\sqrt{4-r^2}}.$$

We can then calculate the area as

$$\int_0^{2\pi}\int_0^{\sqrt{3}} \frac{2r\,dr\,d\theta}{\sqrt{4-r^2}} = 4\pi.$$

I'll leave it to you to check the calculation.